CORE 1

1 Linear Graphs and Equations

\[ y = mx + c \]

\[
\text{gradient} = \frac{\text{increase in } y}{\text{increase in } x}
\]

y intercept

Gradient Facts

- Lines that have the same gradient are PARALLEL
- If 2 lines are PERPENDICULAR then \( m_1 \times m_2 = -1 \) or \( m_2 = -\frac{1}{m_1} \)

\[ e.g. \ 2y = 4x - 8 \]

\[ y = 2x - 4 \quad \text{gradient} = 2 \]

\[ \text{gradient of perpendicular line} = -\frac{1}{2} \]

Finding the equation of a straight line

\[ e.g. \text{Find the equation of the line which passes through (2,3) and (4,8)} \]

\[
\text{GRADIENT} = \frac{y_1 - y_2}{x_1 - x_2}
\]

\[
\begin{align*}
\text{GRADIENT} &= \frac{3 - 8}{2 - 4} \\
&= \frac{-5}{-2} = \frac{5}{2}
\end{align*}
\]

Method 1

\[ y - y_1 = m(x - x_1) \]

Using the point (2,3)

\[ y - 3 = \frac{5}{2} (x - 2) \]

\[ y = \frac{5}{2} x - 2 \]

\[ 2y = 5x - 4 \]

Method 2

\[ y = mx + c \]

Using the point (2,3)

\[ 3 = \frac{5}{2} \times 2 + c \]

\[ c = -2 \]

\[ y = \frac{5}{2} x - 2 \]

\[ 2y = 5x - 4 \]

Finding the Mid-Point

Given the points \((x_1, y_1)\) and \((x_2, y_2)\) the midpoint is \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \)

Finding the point of Intersection

Treat the equations of the graphs as simultaneous equations and solve

Find the point of intersection of the graphs \( y = 2x - 7 \) and \( 5x + 3y = 45 \)
Substituting \( y = 2x - 7 \) gives

\[
\begin{align*}
5x + 3(2x - 7) &= 45 \\
5x + 6x - 21 &= 45 \\
11x &= 66 \\
x &= 6
\end{align*}
\]

\( y = 2 \times 6 - 7 = 5 \)

Point of intersection = (6, 5)

2 **Surds**

- A root such as \( \sqrt{5} \) that cannot be written exactly as a fraction is **IRRATIONAL**
- An expression that involves irrational roots is in **SURD FORM** e.g. \( 3\sqrt{5} \)

\[
\begin{align*}
\sqrt{ab} &= \sqrt{a} \times \sqrt{b} \\
\sqrt{\frac{a}{b}} &= \frac{\sqrt{a}}{\sqrt{b}}
\end{align*}
\]

- e.g. \( \sqrt{75} - \sqrt{12} \)

\[
= \sqrt{5 \times 5 \times 3} - \sqrt{2 \times 2 \times 3}
= 5\sqrt{3} - 2\sqrt{3}
= 3\sqrt{3}
\]

- **RATIONALISING THE DENOMINATOR**

\( 3 + \sqrt{2} \) and \( 3 - \sqrt{2} \) is called a pair of **CONJUGATES**

The product of any pair of conjugates is always a rational number

e.g. \( (3 + \sqrt{2})(3 - \sqrt{2}) = 9 - 3\sqrt{2} + 3\sqrt{2} - 2 \)

\( = 7 \)

Rationalise the denominator of \( \frac{2}{1 - \sqrt{5}} \)

\[
\begin{align*}
\frac{2}{1 - \sqrt{5}} \times \frac{1 + \sqrt{5}}{1 + \sqrt{5}} &= \frac{2 + 2\sqrt{5}}{1 - 5} \\
&= \frac{2 + 2\sqrt{5}}{-4} \\
&= -\frac{1 - \sqrt{5}}{2}
\end{align*}
\]

3. **Quadratic Graphs and Equations**

Solution of quadratic equations

- **Factorisation**

\[
\begin{align*}
x^2 - 3x - 4 &= 0 \\
(x + 1)(x - 4) &= 0 \\
x &= -1 \text{ or } x = 4
\end{align*}
\]
Completing the square

\[ x^2 - 4x - 3 = 0 \]

\[ (x - 2)^2 - (2)^2 - 3 = 0 \]

\[ (x - 2)^2 - 7 = 0 \]

\[ x - 2 = \pm \sqrt{7} \]

\[ x = 2 + \sqrt{7} \text{ or } x = 2 - \sqrt{7} \]

Using the formula to solve \( ax^2 + bx + c = 0 \)

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

E.g. Solve \( x^2 - 4x - 3 = 0 \)

\[ x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times (-3)}}{2 \times 1} \]

\[ = \frac{4 \pm \sqrt{28}}{2} \]

\[ = 2 \pm \sqrt{7} \]

The graph of \( y = ax^2 + bx + c \) crosses the \( y \)-axis at \( y = c \)
If \( b^2 - 4ac > 0 \) there are 2 real distinct roots
If \( b^2 - 4ac = 0 \) there is one repeated root
If \( b^2 - 4ac < 0 \) there are no real roots

Graphs of Quadratic Functions

- The graph of any quadratic expression in \( x \) is called a PARABOLA

- The graph of \( y = k(x - p)^2 \) is a TRANSLATION of the graph \( y = kx^2 \)

In VECTOR notation this translation can be described as \( \begin{bmatrix} p \\ q \end{bmatrix} \)

The equation can also be written as \( y = k(x - p)^2 + q \)

The VERTEX of the graph is \( (p, q) \)

The LINE OF SYMMETRY is \( x = p \)
4 Simultaneous Equations

- Simultaneous equations can be solved by substitution to eliminate one of the variables

Solve the simultaneous equations \( y - 2x = 7 \) and \( x^2 + xy + 2 = 0 \)

\[
y = 7 + 2x
\]

so \( x^2 + x(7 + 2x) + 2 = 0 \)

\[
3x^2 + 7x + 2 = 0
\]

\[
(3x + 1)(x + 2) = 0
\]

\[
x = -\frac{1}{3} \quad y = 6\frac{1}{3} \quad \text{or} \quad x = -2 \quad y = 3
\]

- A pair of simultaneous equations can be represented as graphs and the solutions interpreted as points of intersection.

If they lead to a quadratic equation then the DISCRIMINANT tells you the geometrical relationship between the graphs of the functions

\[
b^2 - 4ac < 0 \quad \text{no points of intersection}
\]

\[
b^2 - 4ac = 0 \quad \text{1 point of intersection}
\]

\[
b^2 - 4ac > 0 \quad \text{2 points of intersection}
\]

5 Inequalities

- Linear Inequality

  - Can be solved like a linear equation except
  
  Multiplying or dividing by a negative value reverses the direction of the inequality sign

  e.g. Solve \(-3x + 10 \leq 4\)

\[
-3x + 10 \leq 4
\]

\[
-3x \leq -6
\]

\[
x \geq 2
\]

- Quadratic Inequality

  - Can be solved by either a graphical or algebraic approach.

  e.g. solve the inequality \( x^2 + 4x - 5 < 0 \)

**Algebraic** \( x^2 + 4x - 5 < 0 \) factorising gives \((x + 5)(x - 1) < 0\)

Using a sign diagram

\[
x + 5 \quad - - - - 0 + + + + + + + +
\]

\[
x - 1 \quad - - - - - - - 0 + + +
\]

\[
(x + 5)(x - 1) \quad + + + 0 - - - 0 + + +
\]

The product is negative for \(-5 < x < 1\)

**Graphical**

The curve lies below the x-axis for \(-5 < x < 1\)
6 Polynomials

Translation of graphs

To find the equation of a curve after a translation of \[ \begin{bmatrix} p \\ q \end{bmatrix} \] replace \( x \) with \( (x-p) \) and replace \( y \) with \( (y - p) \)

e.g The graph of \( y = x^3 \) is translated by \[ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \]

The equation for the new graph is \( y = (x - 3)^3 - 1 \)

Polynomial Functions

A polynomial is an expression which can be written in the form \( a + bx + cx^2 + dx^3 + ex^4 + fx^5 \) (\( a, b, c, \ldots \) are constants)

- Polynomials can be divided to give a QUOTIENT and REMAINDER

\[
\begin{array}{c}
x^2 + 3x + 7 \\
x + 2 \end{array} \quad \begin{array}{c}
x^3 - x^2 + x + 15 \\
x^3 + 2x^2 \\
-3x^2 + x \\
-3x^2 - 6x \\
7x + 15 \\
7x + 14 \\
1
\end{array}
\]

Quoient

Remainder

- REMAINDER THEOREM
  When \( P(x) \) is divided by \( (x - a) \) the remainder is \( P(a) \)

- FACTOR THEOREM
  If \( P(a) = 0 \) then \( (x - a) \) is a factor of \( P(x) \)

e.g. The polynomial \( f(x) = hx^3 - 10x^2 + kx + 26 \) has a factor of \( (x - 2) \)
  When the polynomial is divided by \( (x+1) \) the remainder is 15.
  Find the values of \( h \) and \( k \).

Using the factor theorem \( f(2) = 0 \)
\[
8h - 40 + 2k + 26 = 0 \\
8h + 2k = 14
\]

Using the remainder theorem \( f(-1) = 15 \)
\[
-h - 10 - k + 26 = 14 \\
h + k = 2
\]

Solving simultaneously \( k = 2 - h \)
\[
8h + 2(2 - h) = 14 \\
6h + 4 = 14
\]
A circle with centre \((0,0)\) and radius \(r\) has the equation \(x^2+y^2=r^2\)

A circle with centre \((a,b)\) and radius \(r\) has the equation \((x-a)^2+(y-b)^2=r^2\)

\[\text{e.g. A circle has equation } x^2+y^2+2x-6y=0\]

Find the radius of the circle and the coordinates of its centre.

\[
x^2+2x+y^2-6y=0
\]

\[
(x+1)^2-1+(y-3)^2-9=0
\]

\[
(x+1)^2+(y-3)^2=10
\]

Centre \((1,3)\) radius = \(\sqrt{10}\)

A line from the centre of a circle to where a tangent touches the circle is perpendicular to the tangent. A perpendicular to a tangent is called a NORMAL.

\[\text{e.g. C(-2,1) is the centre of a circle and S(-4,5) is a point on the circumference. Find the equations of the normal and the tangent to the circle at S.}\]

\[
\text{Gradient of SC is } \frac{1-5}{-2(4)} = -\frac{4}{2} = -2
\]

\[
\text{Equation of SC } y = -2x - 3
\]

\[
\text{Gradient of the tangent } = -\frac{1}{2} = \frac{1}{2}
\]

\[
\text{Equation of } y = \frac{1}{2}x + 7
\]

Solving simultaneously the equations of a line and a circle results in a quadratic equation.

\[
b^2 - 4ac > 0 \quad \text{the line intersects the circle}
\]

\[
b^2 - 4ac = 0 \quad \text{the line is a tangent to the circle}
\]

\[
b^2 - 4ac < 0 \quad \text{the line fails to meet the circle}
\]

8 **Rates of Change**

The gradient of a curve is defined as the gradient of the tangent

Gradient is denoted \(\frac{dy}{dx}\) if \(y\) is given as a function of \(x\)

Gradient is denoted by \(f'(x)\) if the function is given as \(f(x)\)

The process of finding \(\frac{dy}{dx}\) or \(f'(x)\) is known as DIFFERENTIATING

**Derivatives**

\[
f(x) = x^n \quad f'(x) = nx^{n-1}
\]

\[
f(x) = a \quad f'(x) = 0
\]
\[ y = x^3 + 4x^2 - 3x + 6 \]

\[ \frac{dy}{dx} = 3x^2 + 8x - 3 \]

9 Using Differentiation

- If the value of \( \frac{dy}{dx} \) is positive at \( x = a \), then \( y \) is increasing at \( x = a \)
- If the value of \( \frac{dy}{dx} \) is negative at \( x = a \), then \( y \) is decreasing at \( x = a \)
- Points where \( \frac{dy}{dx} = 0 \) are called stationary points

Minimum and Maximum Points (Stationary Points)

Stationary points can be investigated

- by calculating the gradients close to the point (see above)
- by differentiating again to find \( \frac{d^2y}{dx^2} \) or \( f''(x) \)

\[ \frac{d}{dx} \frac{d^2y}{dx^2} < 0 \text{ then the point is a local maximum} \]
\[ \frac{d}{dx} \frac{d^2y}{dx^2} > 0 \text{ then the point is a local minimum} \]

Optimisation Problems

Optimisation means getting the best result. It might mean maximising (e.g. profit) or minimising (e.g. costs)

10 Integration

- Integration is the reverse of differentiation
\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \]

Constant of integration

e.g. Given that
\[ f'(x) = 8x^3 - 6x \] and that \( f(2) = 9 \) find \( f(x) \)

\[ f(x) = \int 8x^3 - 6x \, dx \]
\[ = \frac{8x^4}{4} - \frac{6x^2}{2} + c \]
\[ = 2x^4 - 3x^2 + c \]

To find \( c \) use \( f(2) = 9 \)

\[ 32 - 12 + c = 9 \]
\[ c = -11 \]

So \( f(x) = 2x^4 - 3x^2 - 11 \)

11 Area Under a Graph

- The area under the graph of \( y = f(x) \) between \( x = a \) and \( x = b \) is found by evaluating the definite integral

\[ \int_a^b f(x) \, dx \]

e.g. Calculate the area under the graph of \( y = 4x - x^3 \) between the lines \( x = 0 \) and \( x = 2 \)

\[ \int_0^2 4x - x^3 \, dx = \]
\[ = 2x^2 - \frac{x^4}{4} \]
\[ = (8 - 4) - (0 - 0) \]
\[ = 4 \]

- An area BELOW the x–axis has a NEGATIVE VALUE