A LEVEL PURE MATHS REVISON NOTES

1 ALGEBRA AND FUNCTIONS

a) <u>INDICES</u>

Rules to learn :

$x^a \times x^b = x^{a+b}$ $\quad x^a \div x^b = x^{a-b}$ $\quad (x^a)^b = x^{ab}$ $\quad x^{-a} = \frac{1}{x^a}$ $\quad x^{\frac{n}{m}} = \sqrt[m]{x^n} = \left(\sqrt[m]{x}\right)^n$

Simplify $2x(x-y)^{\frac{3}{2}} + 3(x-y)^{\frac{1}{2}}$
$= (x-y)^{\frac{1}{2}}(2x(x-y)+3))$
$= (x-y)^{\frac{1}{2}}(2x^2 - 2xy + 3)$

Solve $3^{2x} \times 25^x = 15$
$(3 \times 5)^{2x} = 15^1$
$2x = 1$
$x = \frac{1}{2}$

b) <u>SURDS</u>

- A root such as $\sqrt{3}$ that cannot be written as a fraction is IRRATIONAL
- An expression that involves irrational roots is in SURD FORM
- RATIONALISING THE DENOMINATOR is removing the surd from the denominator (multiply by the conjugate)

Simplify
$\sqrt{75} - \sqrt{12}$
$= \sqrt{5 \times 5 \times 3} - \sqrt{2 \times 2 \times 3}$
$= 5\sqrt{3} - 2\sqrt{3}$
$= 3\sqrt{3}$

Rationalise the denominator $\frac{2}{2-\sqrt{3}}$
$= \frac{2}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$
$= 4 + 2\sqrt{3}$

The conjugate of the denominator $2 - \sqrt{3}$ is $2 + \sqrt{3}$ so that
$(2 - \sqrt{3})(2 + \sqrt{3})$
$= 2^2 - \sqrt{3}^2$
$= 1$

c) <u>QUADRATIC EQUATIONS AND GRAPHS</u>

Factorising – identifying the roots of the equation $ax^2 + bx + c = 0$

- Look for the **difference of 2 squares** $x^2 - a^2 = (x+a)(x-a)$ or $(ax)^2 - b^2 = (ax+b)(ax-b)$
- Look for the perfect square $x^2 + 2ax + a^2 = (x+a)^2$
- Look out for equations which can be transformed into quadratic equations

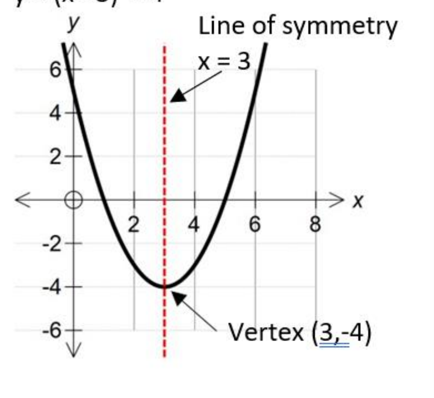
Solve $x + 1 - \frac{12}{x} = 0$
$x^2 + x - 12 = 0$
$(x+4)(x-3) = 0$
$x = -4 \quad x = 3$

Solve $6x^4 - 7x^2 + 2 = 0$
Let $z = x^2$ $\quad 6z^2 - 7z + 2 = 0$
$(2z-1)(3z-2) = 0$
$z = \frac{1}{2} \quad x = \pm\sqrt{\frac{1}{2}} \qquad z = \frac{2}{3} \quad x = \pm\sqrt{\frac{2}{3}}$

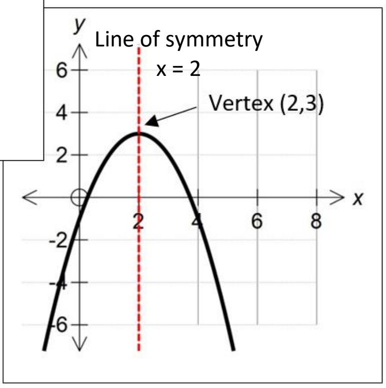
Completing the square – identifying the vertex and line of symmetry

$y = (x+a)^2 + b$ vertex at $(-a, b)$ line of symmetry as equation $x = -a$

$y = (x-3)^2 - 4$



Sketch the graph of
$y = 4x - x^2 - 1$
$y = -(x^2 - 4x) - 1$
$y = -((x-2)^2 - 4) - 1$
$y = -(x-2)^2 + 3$



Quadratic formula (and the DISCRIMINANT)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ for solving $ax^2 + bx + c = 0$

The **DISCRIMINANT $b^2 - 4ac$** can be used to identify the number of roots

$b^2 - 4ac > 0$ there are 2 real distinct roots (graph crosses the x-axis twice)

$b^2 - 4ac = 0$ there is a single repeated root (the x-axis is a tangent)

$b^2 - 4ac < 0$ there are no real roots (the graph does not touch the x-axis)

d) <u>SIMULTANEOUS EQUATIONS</u>

Solving by elimination

$3x - 2y = 19 \quad \times 3 \quad 9x - \mathbf{6y} = 57$

$2x - 3y = 21 \quad \times 2 \quad \underline{4x - \mathbf{6y} = 42}$

$5x - 0y = 15 \qquad x = 3 \quad (9 - 2y = 19) \qquad y = -5$

Solving by substitution

$x + y = 1 \quad (y = 1 - x)$

$x^2 + y^2 = 25 \qquad x^2 + (1 - x)^2 = 25$

$2x^2 - 2x - 24 = 0$

$2(x - 4)(x + 3) = 0 \qquad x = 4 \quad y = -3 \qquad x = -3 \quad y = 4$

If you end up with a quadratic equation when solving simultaneously the discriminant can be used to determine the relationship between the graphs

If $b^2 - 4ac > 0$ the graphs intersect at 2 distinct points

$b^2 - 4ac = 0$ the graphs intersect at 1 point (or tangent)

$b^2 - 4ac < 0$ the graphs do not intersect

e) <u>INQUALITIES</u>

Linear Inequality - solve using the same method as solving a linear equation but remember to reverse the inequality if you multiply or divide by a negative number

Quadratic Inequality – always a good idea to sketch a graph

$\leq \ \geq$ plot the graph as a solid line or curve

$< \ >$ plot as a dotted/dashed line or curve

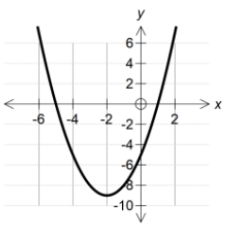
If you are unsure of which area to shade pick a point in one of the regions and check the inequalities using the coordinates of the point

Solve **$x^2 + 4x - 5 < 0$**

$x^2 + 4x - 5 = 0$

$(x - 1)(x + 5) = 0$

$x = 1 \quad x = -5$



$x^2 + 4x - 5 < 0$

$-5 < x < 1$

which can be written as

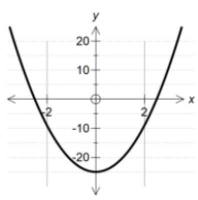
$\{x : x > -5\} \cap \{x : x < 1\}$

Solve **$4x^2 - 25 \geq 0$**

$4x^2 - 25 = 0$

$(2x - 5)(2x + 5) = 0$

$x = \frac{5}{2} \quad x = -\frac{5}{2}$



$4x^2 - 25 \geq 0$

$x \leq -\frac{5}{2}$ or $x \geq \frac{5}{2}$

which can be written as

$\{x : x \leq -\frac{5}{2}\} \cup \{x : x \geq \frac{5}{2}\}$

f) <u>POLYNOMIALS</u>

- A polynomial is an expression which can be written in the form $ax^n + bx^{n-1} + cx^{n-2} + \ldots$ where a,b, c are constants and n is a positive integer.
- The **order** of the polynomial is the highest power of x in the polynomial
- Polynomials can be divided to give a **Quotient** and **Remainder**

Divide $x^3 - x^2 + x + 15$ by $x + 2$

$$\begin{array}{rrrrr}
 & x^2 & -3x & +7 & \leftarrow \text{Quotient} \\
\cline{2-5}
x+2 \,\big) & x^3 \;\; -x^2 & +x & +15 & \\
 & x^3 \;\; +2x^2 & & & \\
\cline{2-2}
 & -3x^2 & +x & & \\
 & -3x^2 & -6x & & \\
\cline{2-3}
 & & 7x & +15 & \\
 & & 7x & +14 & \\
\cline{3-4}
 & & & 1 & \leftarrow \text{Remainder}
\end{array}$$

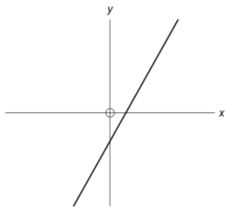
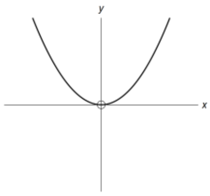
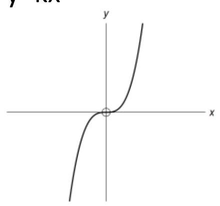
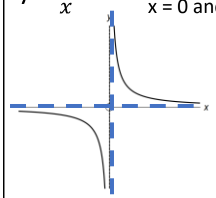
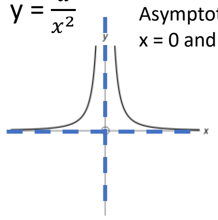
- **Factor Theorem** – If $(x - a)$ is a factor of $f(x)$ then $f(a) = 0$ and is root of the equation $f(x) = 0$

Show that $(x - 3)$ is a factor of $x^3 - 19x + 30 = 0$
$f(x) = x^3 - 19x + 30$
$f(3) = 3^3 - 19 \times 3 + 20$
$\quad = 0$
$f(3) = 0$ so $x - 3$ is a factor of $f(x)$

g) GRAPHS OF FUNCTIONS

Sketching Graphs

- Identify where the graph crossed the y-axis (x = 0)
- Identify where the graph crossed the x-axis (y = 0)
- Identify any asymptotes and plot with a dashed line

y=mx + c	$y = kx^2$	$y=kx^3$	$y = \frac{a}{x}$ Asymptotes at x = 0 and y = 0	$y = \frac{a}{x^2}$ Asymptotes at x = 0 and y = 0
	 y is proportional to x^2	 y is proportional to x^3	 y is proportional to $\frac{k}{x}$	 y is proportional to $\frac{a}{x^2}$

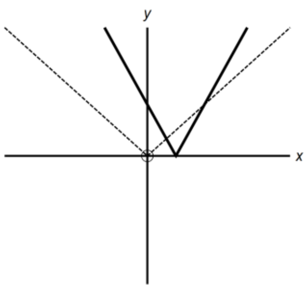
Modulus Graphs

- $|x|$ is the 'modulus of x' or the absolute value $|2|=2$ $|-2|= 2$
- To sketch the graph of $y = |f(x)|$ sketch $y = f(x)$ and take any part of the graph which is below the x-axis and reflect it in the x-axis

Solve $|2x - 4|<|x|$

$2x - 4 = x$ $\qquad$ $2x - 4 = -x$
$x = 4$ $\qquad$ $3x = 4$
$\qquad\qquad x = \frac{4}{3}$

$\frac{4}{3} < x < 4$



h) FUNCTIONS

- A function is a rule which generates exactly ONE OUTPUT for EVERY INPUT
- **DOMAIN** – defines the set of the values that can be 'put into' the function f(x) = $\sqrt{x}$ domain x ≥ 0
- **RANGE** – defines the set of values 'output' by the function – make sure it is defined in terms of f(x) and not x
  $f : x \mapsto x^2 \quad x \in \mathbb{R}$ means an input a is converted to a^2 where the input 'a' can be any real number
  $Range\ f(x) \geq 0$

- **INVERSE FUNCTION** denoted by $f^{-1}(x)$
  The domain of $f^{-1}(x)$ is the range of f(x)
  The range of $f^{-1}(x)$ is the domain of f(x)

  Using the same scale on the x and y axis the graphs of a function and it's inverse have **reflection symmetry** in the line **y = x**

$f(x) = \frac{3}{x+2}$ find $f^{-1}(x)$

$y = \frac{3}{x+2}$

$x = \frac{3}{y} - 2$

$f^{-1}(x) = \frac{3}{x} - 2$

- **COMPOSITE FUNCTIONS**
  The function **gf**(x) is a composite function which tells you 'to do' **f** first and then use the output in **g**

f(x) = 4x g(x) = x^2 – 1

fg(x) = $4(x^2 - 1)$
$= 4x^2 - 4$

gf(x) = $(4x)^2 - 1$
$= 16x^2 - 1$

i) TRANSFORMING GRAPHS

Translation

To find the equation of a graph after a translation of $\begin{bmatrix} a \\ b \end{bmatrix}$ replace x by (x – a) and y by (y – b)

y = f(x -a) + b

The graph of y = x^2 -1 is translated by $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Find the equation of the resulting graph.

$(y + 2) = (x - 3)^2 - 1$

$y = x^2 - 6x + 6$

Reflection

Reflection in the x-axis replace y with -y y = -f(x)

Reflection in the y-axis replace x with -x y = f(-x)

Stretch

Stretch in the y-direction by scale factor a y = af(x)

Stretch on the x-direction by scale factor $\frac{1}{a}$ y = f(ax)

Combining Transformations

Take care with the order in which the transformations are carried out.

The graph of y = x^2 **is translated by** $\begin{bmatrix} \mathbf{3} \\ \mathbf{0} \end{bmatrix}$ and then **reflected in the y axis**. Find the equation of the resulting graph

Translation $y = (x - 3)^2$
$= x^2 - 6x + 9$

Reflection $y = (-x)^2 - 6(-x) + 9$
$\mathbf{= x^2 + 6x + 9}$

The graph of y = x^2 is **reflected in the y axis** and then **translated by** $\begin{bmatrix} \mathbf{3} \\ \mathbf{0} \end{bmatrix}$. Find the equation of the resulting graph

Reflection $y = (-x)^2$
$= x^2$

Translation $y = (x - 3)^2$
$\mathbf{= x^2 - 6x + 9}$

j) PARTIAL FRACTIONS

Any proper algebraic fractions with a denominator that is a product of linear factors can be written as partial fractions

- Useful for integrating a rational function
- Useful for finding binomial approximations

$$\frac{px+q}{(ax+b)(cx+d)(ex+f)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f} \qquad \frac{px+q}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$$

Express $\frac{5}{(x-2)(x+3)}$ in the form $\frac{A}{x-2} + \frac{B}{x+3}$

$\frac{A}{x-2} + \frac{B}{x+3} \equiv \frac{A(x+3)+B(x-2)}{(x+3)(x-2)}$

$A(x + 3) + B(x - 2) = 5$ $\quad x = 2 \quad 5A = 5 \quad x = -3 \quad -5B = 5$

$A = 1 \quad B = -1$

$$\frac{5}{(x-2)(x+3)} = \frac{1}{x-2} - \frac{1}{x+3}$$

2 COORDINATE GEOMETRY

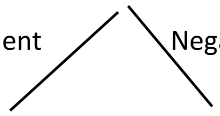
a) Graphs of linear functions

y = mx + c

the line intercepts the y axis at (0, c)

Gradient = $\frac{change\ in\ y}{change\ in\ x}$

Positive gradient



Negative gradient

Finding the equation of a line with gradient m through point (x_1, y_1)

Use the equation $(y - y_1) = m(x - x_1)$

If necessary rearrange to the required form (ax + by = c or y = mx + c)

Parallel and Perpendicular Lines

$y = m_1x + c_1 \qquad y = m_2x + c_2$

If $\mathbf{m_1 = m_2}$ then the lines are **PARALLEL**

If $\mathbf{m_1 \times m_2 = -1}$ then the lines are **PERPENDICULAR**

Find the equation of the line perpendicular to the line y – 2x = 7 passing through point (4, -6)

Gradient of y – 2x = 7 is 2 (y = 2x + 7)

Gradient of the perpendicular line = - ½ (2 × -½ = -1)

Equation of the line with gradient –½ passing through (4, -6)

(y + 6) = -½(x – 4)

2y + 12 = 4 – x

x + 2y = - 8

Finding the mid-point of the line segment joining (a,b) and (c,d)

Mid-point = $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

Calculating the length of a line segment joining (a,b) and (c,d)

Length = $\sqrt{(c-a)^2 + (d-b)^2}$

b) Circles

A circle with centre (0,0) and radius r has the equations $x^2 + y^2 = r^2$

A circle with centre (a,b) and radius r is given by $(x - a)^2 + (y - b)^2 = r^2$

Finding the centre and the radius (completing the square for x and y)

Find the centre and radius of the circle $x^2 + y^2 + 2x – 4y – 4 = 0$

$x^2 + 2x + y^2 – 4y – 4 = 0$

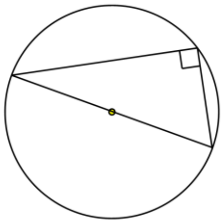
$(x + 1)^2 – 1 + (y – 2)^2 – 4 – 4 = 0$

$(x + 1)^2 + (y – 2)^2 = 3^2$

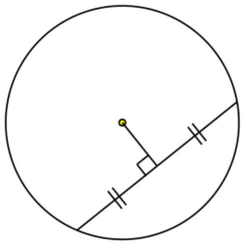
Centre (-1, 2) Radius = 3

The following circle properties might be useful

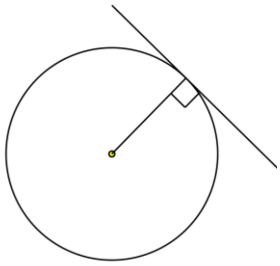
Angle in a semi-circle is a right angle



The perpendicular from the centre to a chord bisects the chord



The tangent to a circle is perpendicular to the radius



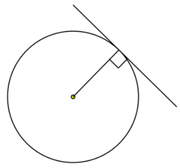
Finding the equation of a tangent to a circle at point (a,b)

The gradient of the tangent at (a,b) is perpendicular to the gradient of the radius which meets the circumference at (a, b)

Find equation of the tangent to the circle $x^2 + y^2 - 2x - 2y – 23 = 0$ at the point (5,4)

$(x - 1)^2 + (y – 1)^2 – 25 = 0$

Centre of the circle (1,1)



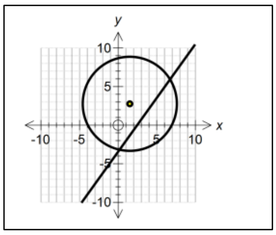
Gradient of radius $= \frac{4-1}{5-1} = \frac{3}{4}$ Gradient of tangent $= -\frac{4}{3}$

Equation of the tangent $(y – 4) = -\frac{4}{3}(x – 5)$ $3y – 12 = 20 - 4x$

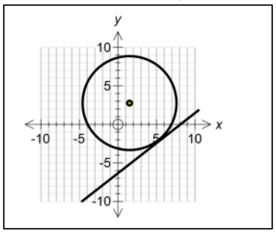
$4x + 3y = 32$

Lines and circles Solving simultaneously to investigate the relationship between a line and a circle will result in a quadratic equation. **Use the discriminant to determine the relationship between the line and the circle**

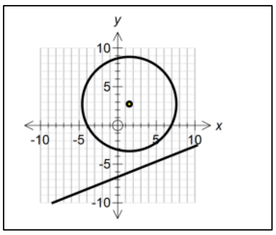
$b^2 – 4ac > 0$



$b^2 – 4ac = 0$ (tangent)



$b^2 – 4ac < 0$



c) Parametric Equations

- Two equations that separately define the x and y coordinates of a graph in terms of a third variable
- The third variable is called the **parameter**
- To convert a pair of **parametric equations** to a **cartesian equation** you need to eliminate the parameter (you may need to use trig identities if the parametric equations involve trig functions)

Find the cartesian equation of the curve given by the parametric equations given by $x = cos\theta \quad y = sin2\theta$
$y = sin2\theta$
$y = 2sin\theta cos\theta \qquad y^2 = 4sin^2\theta cos^2\theta$
$= 4(1 - cos^2\theta)cos^2\theta$
$y^2 = 4(1 - x^2)x^2$

3. SEQUENCES AND SERIES

a) Binomial

Expansion of $(1 + x)^n \quad |x| < 1 \quad n \in \mathbb{Q}$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}x^3 \ldots\ldots\ldots\ldots + nx^{n-1} + x^n$$

Use the binomial expansion to write down the first four terms of $\frac{1}{(2-3x)^2}$

$$2^{-2}\left(1 - \frac{3}{2}x\right)^{-2} = 2^{-2}(1 + -2 \times (-\tfrac{3}{2}x) + \frac{-2\times-3}{1\times2}\left(-\frac{3}{2}x\right)^2 + \frac{-2\times-3\times-4}{1\times2\times3}(-\tfrac{3}{2}x)^3$$
$$= \frac{1}{4}(1 + 3x + \frac{27}{4}x^2 + \frac{27}{2}x^3)$$
$$= \frac{1}{4} + \frac{3}{4}x + \frac{27}{16}x^2 + \frac{27}{8}x^3$$

Expansion of $(a + b)^n \quad n \in \mathbb{Z}^+$

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\times2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\times2\times3}a^{n-3}b^3 \ldots\ldots\ldots\ldots + nab^{n-1} + b^n$$

Find the coefficient of the x^3 term in the expansion of $(2 + 3x)^9$

$(3x)^3$ must have 2^6 as part of the coefficient ($^{3+6=9}$)

$\frac{9\times8\times7}{1\times2\times3} \times 2^6 \times (3x)^3$ = 145152 (x^3)

b) Sequences

- An inductive definition defines a sequence by giving the first term and a rule to find the next term(s)
  $u_{n+1} = f(u_n) \qquad u_1 = a$

Find the first 3 terms of a sequence defined by $u_{n+1} = 2u_n + 1 \qquad u_1 = 2$
$u_1 = 2 \quad u_2 = 2 \times 2 + 1 \qquad u_3 = 2 \times 5 + 1$
= 5 = 11

- An **increasing** sequence is one where $u_{n+1} > u_n$ for all n
- An **decreasing** sequence is one where $u_{n+1} < u_n$ for all n
- A sequence may converge to a **limit L** $u_{n+1} = f(u_n)$ as $n \to \infty \qquad u_{n+1} = u_n = L$

The sequence defined by $u_{n+1} = 0.2u_n + 2 \quad u_1 = 3$ converges to a limit L . Find L
L = 0.2L + 2 0.8L = 2 L = 2.5

- A periodic sequence repeats itself over a fixed interval
  $u_{n+a} = u_n$ for all n for a constant a which is the period of the sequence

c) Sigma Notation – sum of

$\sum_{r=1}^{6}(r^2 + 1)$ = (1^2+1) + (2^2+1) + (3^2+1) + (4^2+1) + (5^2+1) + (6^2+1)
= 2 + 5 + 10 + 17 + 26 + 37
= 97

Staring with the 1st term r = 1
Ending with the 6th term r = 6

d) Arithmetic sequences and series

- Each term is found by adding a fixed constant (**common difference d)** to the previous term
- The first term is **a** giving the sequence a , a + d, a + 2d, a + 3d…… where $u_n = a + (n-1)d$
- The sum of the first n terms can be found using:

$$S_n = \frac{n}{2}(2a + (n-1)d) \quad \text{or} \quad S_n = \frac{n}{2}(a + l) \quad \text{where } l \text{ is the last term}$$

e) Geometric sequence and series

- Each term is found by multiplying the previous term by a fixed constant (**common ratio r)**
- The first term is **a** giving the sequence a, ar, ar^2, ar^3, ar^4…
- The sum of the first n terms can be found using

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1} \qquad S_\infty = \frac{a}{1-r} \quad |r| < 1$$

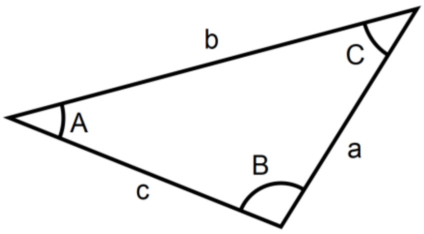
4. TRIGONOMETRY

MAKE SURE YOU KNOW AND CAN USE THE FOLLOWING FROM GCSE

$$Area = \frac{1}{2}absinC$$

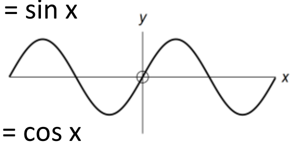
$$\frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC} \quad \text{or} \quad \frac{SinA}{a} = \frac{SinB}{b} = \frac{SinC}{c}$$

$$a^2 = b^2 + c^2 - 2bcCosA$$

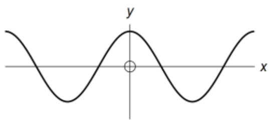


	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
Sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	1
Tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	---	0

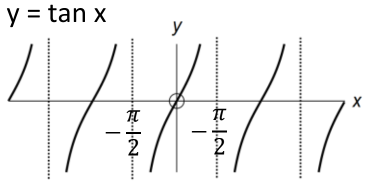
y = sin x



y = cos x

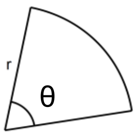


y = tan x



a) Radians 2π radians = 360° π radians = 180°

- You MUST work in radians if you are integrating or differentiating trig functions
- For an angle at the centre of a sector of θ **radians**



Arc Length = rθ

Area of the sector = ½r^2θ

b) Small angle approximations (θ in radians)

$$sin\theta \approx \theta \qquad cos\theta \approx 1 - \frac{\theta^2}{2} \qquad tan\theta \approx \theta$$

When θ is small show that $\frac{cos\theta}{sin\theta}$ can be written as $\frac{2-\theta^2}{2\theta}$

$$\left(1 - \frac{\theta^2}{2}\right) \div \theta$$

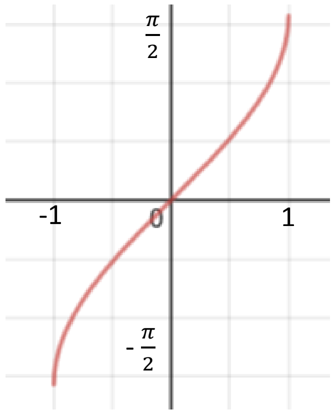
$$= \frac{2-\theta^2}{2} \div \theta$$

$$= \frac{2-\theta^2}{2\theta}$$

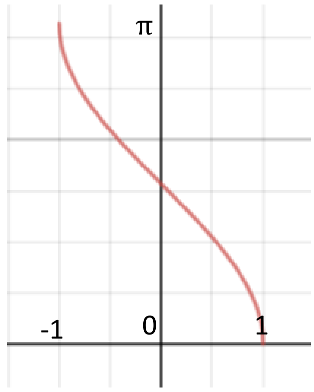
c) <u>Inverse Functions</u> ($\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$)

By definition a function must be one-to-one which leads to restricted domains for the inverse trig functions

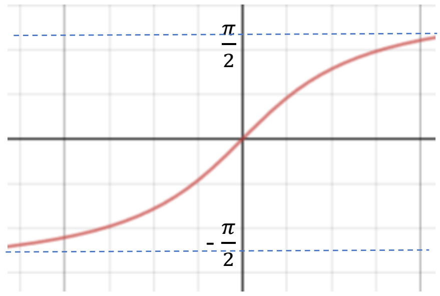
y = $\sin^{-1}$x (arcsin x)
Domain : $-1 \le x \le 1$



y = $\cos^{-1}$x (arccos x)
Domain : $-1 \le x \le 1$



y = $\tan^{-1}$x (arctan x)
Domain $x \in \mathbb{R}$



d) <u>Reciprocal Trig Functions and identities</u> (derived from $sin^2x + cos^2x = 1$)

$\sec x = \frac{1}{\cos x}$ $\qquad cosec\ x = \frac{1}{\sin x}$ $\qquad cot\ x = \frac{1}{\tan x}\ \left(\frac{\cos x}{\sin x}\right)$

$1 + tan^2x = sec^2x$ $\qquad 1 + cot^2x = cosec^2x$

Solve for 0° < θ < 360° the equation $sec^4\theta - tan^4\theta = 2$

$sec^2x = 1 + tan^2x$

$sec^4x = 1 + 2tan^2x + tan^4x$ $\qquad 1 + 2tan^2x + tan^4x - tan^4x = 2$

$2tan^2x = 1$ $\qquad \tan x = \pm\sqrt{\frac{1}{2}}$ $\qquad$ x = 35.3°, 145°, 215°, 325° $\quad$ 3 s.f.

e) <u>Double angle and addition formulae</u>

The addition formulae are given the formula booklet

Make sure you can use these to derive :

DOUBLE ANGLE FORMULAE

sin 2A = 2 sinAcosA

cos 2A = $\cos^2$A – $\sin^2$A

= $2\cos^2$A – 1

= 1 - $2\sin^2$A

Tan 2A = $\frac{2\tan A}{1 - tan^2A}$

$\sin(A \pm B) = sinA\ cosB \pm cosA sinB$

$\cos(A \pm B) = cosA\ cosB \mp sinA sinB$

$\tan(A \pm B) = \frac{tanA \pm tanB}{1 \mp tanA tanB}$

- Useful to solve equations
- cos 2A often used to integrate trig functions involving $\sin^2$x or $\cos^2$x

EXPRESSING IN THE FORM $rsin(\theta \pm \alpha)$ and $rcos(\theta \pm \alpha)$

- Useful in solving equations asin θ + bcos θ = 0
- Useful in finding minimum/maximum values of $acos\theta \pm bsin\theta$ and $asin\theta \pm bcos\theta$

Find the maximum value of the expression 2sinx + 3cosx and the value of x where this occurs (x < 180°)

2sinx + 3cosx = Rsin(x + α) $\quad$ (Rsin x cos α + Rcos x sin α) $\quad$ rcos α = 2 $\quad$ rsin α = 3

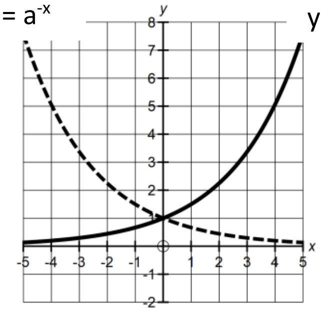
R = $\sqrt{2^2 + 3^2}$ $\qquad$ tanα = $\frac{3}{2}$

= $\sqrt{13}$ $\qquad$ α = 56.3° $\qquad$ 2sinx + 3cosx = $\sqrt{13}\sin(x + 56.3°)$

Max value = $\sqrt{13}$ occurs when $\sin(x + 56.3°)$ = 1

X = 33.7°

5 LOGARITHMS AND EXPONENTIALS



- A function of the form $y = a^x$ is an exponential function
- The graph of $y = a^x$ is positive for all values of x and passes through (0,1)
- A logarithm is the inverse of an exponential function

$y = a^x \qquad x = \log_a y$

Logarithms – rules to learn

$\log_a a = 1 \qquad \log_a 1 = 0 \qquad \log_a a^x = x \qquad a^{\log_a x} = x$

$\log_a m + \log_a n = \log_a mn \qquad \log_a m - \log_a n = \log_a\left(\frac{m}{n}\right) \qquad k\log_a m = \log_a m^k$

Write the following in the form alog 2 where a is an integer 3log 2 + 2log 4 – ½log16

Method 1 : $\log 8 + \log 16 - \log 4 = \log\left(\frac{8\times16}{4}\right) = \log 32 = 5\log 2$

Method 2 : $3\log 2 + 4\log 2 - 2\log 2 = 5\log 2$

An equation of the form $a^x = b$ can be solved by taking logs of both sides

a) MODELLING CURVES

Exponential relationships can be changed to a linear form y = mx + c allowing the constants m and c to be 'estimated' from a graph of plotted data

$\mathbf{y = Ax^n} \qquad \log y = \log(Ax^n) \qquad \log y = n\log x + \log A$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad y \;\; = mx \; + \; c$

Plot log y against log x. n is the gradient of the line and log A is the y axis intercept

$\mathbf{y = Ab^x} \qquad \log y = \log(Ab^x) \qquad \log y = x\log b + \log A$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad y \;\; = mx \; + \; c$

Plot log y against x. log b is the gradient of the line and log A is the y axis intercept

V and x are connected by the equation $\mathbf{V = ax^b}$

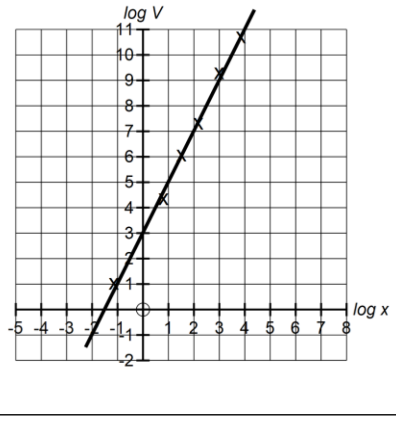
The equation is reduced to linear form by taking logs

$\log V = b\log x + \log a$

(y = mx + c) (log V plotted against log x)

From the graph b = 2

$\log a = 3 \quad a = 10^3$

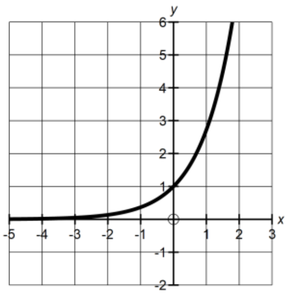


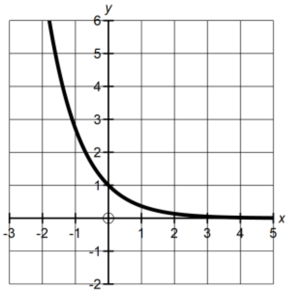
Gradient = 2

Intercept = 3

b) The exponential function y = e^x

Exponential Growth y = e^x **Exponential Decay** y = e^{-x}





The inverse of y = e^x is the **natural logarithm** denoted by **ln x**

Solve $2e^{x-2} = 6$ leaving your answer in exact form

$$e^{x-2} = 3$$
$$\ln(e^{x-2}) = \ln 3$$
$$x - 2 = \ln 3$$
$$x = \ln 3 + 2$$

The rate of growth/decay to find the 'rate of change' you need to differentiate to find the gradient

LEARN THIS

$$y = Ae^{kx} \qquad \frac{dy}{dx} = Ake^{kx}$$

The number of bacteria P in a culture is modelled by
P = $600 + 5e^{0.2t}$ where t is the time in hours from the start of the experiment. Calculate the rate of growth after 5 hours

P = $600 + 15e^{0.2t}$ $\frac{dP}{dt} = 3e^{0.2t}$

t = 5 $\frac{dP}{dt} = 3e^{0.2\times5}$

= 8.2 bacteria per hour

6 DIFFERENTIATION

- The gradient is denoted by $\frac{dy}{dx}$ if y is given as a function of x
- The gradient is denoted by f'(x) is the function is given as f(x)

LEARN THESE

$$y = x^n \quad \frac{dy}{dx} = nx^{n-1} \qquad y = ax^n \quad \frac{dy}{dx} = nax^{n-1} \qquad y = a \quad \frac{dy}{dx} = 0$$

$$y = e^{kx} \quad \frac{dy}{dx} = ke^x \qquad y = lnx \quad \frac{dy}{dx} = \frac{1}{x} \qquad y = a^{kx} \quad \frac{dy}{dx} = (klna)a^{kx}$$

$$y = \sin kx \quad \frac{dy}{dx} = kcos\, kx \qquad y = \cos kx \quad \frac{dy}{dx} = -ksin\, kx \qquad y = \tan kx \quad \frac{dy}{dx} = ksec^2 kx$$

a) Methods of differentiation

CHAIN RULE for differentiating **y = fg(x)** y = f(u) where u = g(x) $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

PRODUCT RULE for differentiating **y = f(x)g(x)** $\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$

QUOTIENT RULE for differentiating **y =** $\frac{f(x)}{g(x)}$ $\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

PARAMETRIC EQUATIONS y = f(t) x = g(t) $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

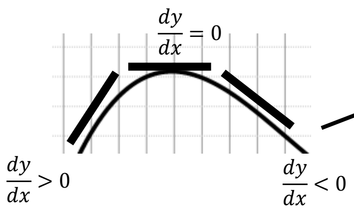
IMPLICIT DIFFERENTIATION- take care as you may need to use the product rule too (xy^2, xy, ysinx)

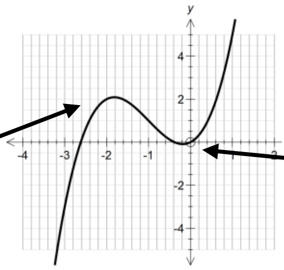
$$\frac{d[f(y)]}{dx} = \frac{d[f(y)]}{dy} \times \frac{dy}{dx}$$

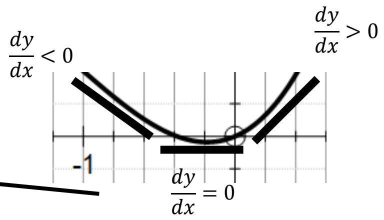
b) <u>Stationary (Turning) Points</u>

- The points where $\frac{dy}{dx} = 0$ are stationary points (turning points/points of inflection) of a graph
- The nature of the turning points can be found by:

Maximum point **Minimum Point**







Minimum if $\frac{d^2y}{dx^2} > 0$

Maximum if $\frac{d^2y}{dx^2} < 0$

Find and determine the nature of the stationary points of the curve $y = 2x^3 - 3x^2 + 18$

$\frac{dy}{dx} = 6x^2 - 6x$ $\frac{dy}{dx} = 0$ at a stationary point

$6x(x - 1) = 0$ Turning points at (0, 18) and (1,17) $\frac{d^2y}{dx^2} = 12x - 6$ $x = 0$ $\frac{d^2y}{dx^2} < 0$ (0,18) is a maximum

$x = 1$ $\frac{d^2y}{dx^2} > 0$ (1,17) is a minimum

Points of inflection occur when $\frac{d^2y}{dx^2} = 0 \quad (f''(x) = 0)$

.........but $\frac{d^2y}{dx^2} = 0$ could also indicate a min or max point

Convex curve : $\frac{d^2y}{dx^2} > 0$ for all values of x in the 'convex section of the curve'

Concave curve : $\frac{d^2y}{dx^2} < 0$ for all values of x in the 'concave section of the curve'

c) <u>Using Differentiation</u>

Tangents and Normals

The gradient of a curve at a given point = gradient of the tangent to the curve at that point
The gradient of the **normal** is perpendicular to the gradient of the tangent that point

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Find the equation of the normal to the curve $y = 8x - x^2$ at the point (2,12)

$\frac{dy}{dx} = 8 - 2x$ Gradient of tangent at (2,12) = 8 – 4 = 4

Gradient of the normal = - ¼ (y - 12) = -¼ (x – 2)

4y + x = 50

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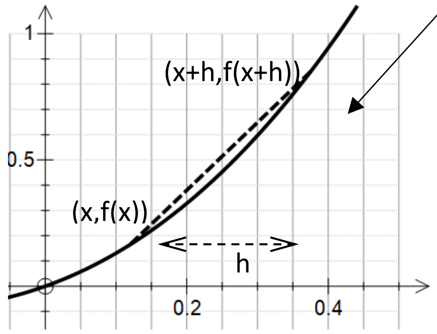
d) Differentiation from first principles



As h approaches zero the gradient of the chord gets closer to being the gradient of the tangent at the point

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

Find from first principles the derivative of $x^3 - 2x + 3$

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{(x+h)^3-2(x+h)+3-(x^3-2x+3)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{x^3+3x^2h+3xh^2+h^3-2x-2h+3-x^3+2x-3)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{3x^2h+3xh^2+h^3-2h}{h}\right)$$
$$= \lim_{h\to 0}(3x^2+3xh+h^2-2)$$
$$= 3x^2 - 2$$

7 INTEGRATION

Integration is the reverse of differentiation

LEARN THESE

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ (c is the constant of integration)

$\int e^{kx}dx = \frac{1}{k}e^{kx} + c$ $\int \frac{1}{x}dx = lnx + c$

$\int \sin kx\,dx = -\frac{1}{k}\cos kx + c$ $\int \cos kx\,dx = \frac{1}{k}\sin kx + c$

a) Methods of Integration

INTEGRATION BY SUBSTITUTION

Transforming a complex integral into a simpler integral using 'u = ' and integrating with respect to u

$\int x\sqrt{1-x^2}\,dx$

Let u = 1 - x^2 $\frac{du}{dx}$ = -2x so dx = $\frac{du}{-2x}$

$$\int x\sqrt{1-x^2}\,dx = \int x\sqrt{u}\,\frac{du}{-2x}$$
$$= -\frac{1}{2}\int u^{\frac{1}{2}}\,du$$
$$= -\frac{1}{3}u^{\frac{3}{2}} + c$$
$$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$

If it is a definite integral it is often easier to calculate the limits in terms of u and substitute these in after integrating

Look for integrals of the form

$\int e^{ax+b}dx$ $\int \cos(ax+b)\,dx$ $\int \frac{1}{ax+b}dx$

Look out for integrals of the form

$$\int f'(x)[f(x)]^n = \frac{1}{n+1}[f(x)]^{n+1} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

INTEGRATION BY PARTS

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Take care in defining u and $\frac{dv}{dx}$

$\int xe^{2x}\,dx \qquad u = x \qquad \frac{dv}{dx} = e^{2x}$

$\int xln\,dx \qquad u = lnx \qquad \frac{dv}{dx} = x$

$\int lnx\,dx \qquad u = \ln x \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x} \qquad v = x$

$\int lnx\,dx = xlnx - \int x\frac{1}{x}\,dx$

$= xlnx - x + c$

PARAMETRIC INTEGRATION

To find the area under a curve defined parametrically use area = $\int y\frac{dx}{dt}dt$

Remember that the limits of the integral must be in terms of t

A curve is defined parametrically by $x = t - 1 \quad y = \frac{4}{t}$ Calculate the area of the region included by the line x= 2, the x-axis and the y-axis.

x = 2 t = 3 x = 0 t = 1

$\frac{dx}{dt} = 1 \qquad \int_1^3 \frac{4}{t}\,dt = [4lnt]_1^3$

$= 4ln3 - 4ln1$

= 4ln 3

b) AREA UNDER A CURVE

The area under a graph can be approximated using rectangle of height y and width dx. The limit as the number of rectangles increases is equal to the definite integral

$$\lim_{n\to\infty}\sum_{i=1}^{n} y_i\delta x = \int_a^b y\,dx$$

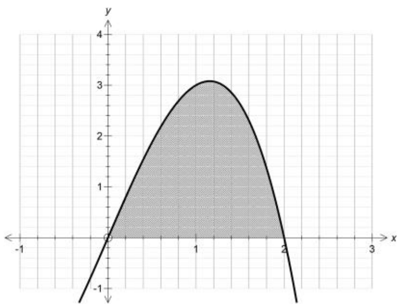
Calculate the area under the graph y = 4x – x^3 between x = 0 and x = 2

$\int_0^2 4x - x^3dx$

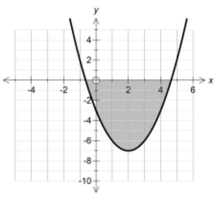
$= \left[2x^2 - \frac{x^4}{4}\right]_0^2$

$= (8-4) - (0-0)$

$= 4$



For an area below the x-axis the integral will result in a **negative value**

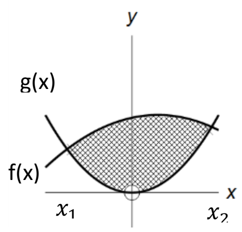


c) AREA BETWEEN 2 CURVES

If no limits are given you need to identify the x coordinates of the points where the curve intersect

Determine which function is ‘above’ the other

$\int_{x_1}^{x_2}[f(x) - g(x)]dx$



d) SOLUTION OF DIFFERENTIAL EQUATIONS

Separating the variables

If you are given the coordinates of a point on the curve a particular solution can be found if not a general solution is needed

Find the general solution for the differential equation

$y\frac{dy}{dx} = xy^2 + 3x$

$y\frac{dy}{dx} = x(y^2 + 3)$

$\int \frac{y}{y^2+3} dy = \int x\, dx$

$\frac{1}{2}\ln|y^2 + 3| = \frac{1}{2}x^2 + c$

8 NUMERICAL METHODS

a) CHANGE OF SIGN – locating a root

For an equations f(x) = 0 , if $f(x_1)$ and $f(x_2)$ have opposite signs and f(x) is a **continuous function** between x_1 and x_2 then a root of the equation lies in the interval $x_1 < x < x_2$

b) STAIRCASE and COWBEB DIAGRAMS

If an **iterative formula** (recurrence relation) of the form $x_{n+1}=f(x_n)$ converges to a limit, the value of the limit is the x-coordinate of the point of intersection of the graphs y = f(x) and y = x

The limit is the solution of the equation f(x) = x

A staircase or cobweb diagram based on the graphs y = f(x) and y = x shows the convergence

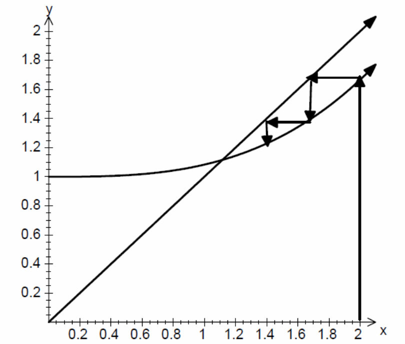
e.g Solve the equation $x^3 - 12x + 12 = 0$

First we will write it in the form x = f(x)

$x^3 + 12 = 12x \Rightarrow \frac{x^3}{12} + 1 = x$

Plotting the graphs $y = \frac{x^3}{12} + 1$ and $y = x$

the solution is the point of intersection of the two graphs.



We can confirm that there is a point of intersection between x = 1 and x = 2 by a change of sign the values are substituted.

Substituting x = 2 into $y = \frac{x^3}{12} + 1$ *gives y = 1.66… (shown on the diagram)*

Substituting x = 1.66…. y = 1.38…

Repeating this the values converge to 1.1157

The solution of $x^3 - 12x + 12 = 0$ *is x = 1.1157*

Use you **ANS** button on your calculator

c) NEWTON-RAPHSON iteration $f(x) = 0 \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

The equation $e^{-2x} - 0.5x = 0$ has a root close to 0.5. Using 0.5 as the first approximation use the Newton-Raphson you find the next approximation

$x_1 = 0.5$ $f(0.5) = e^{-1} - 0.25$

$f'(x) = -2e^{-2x} - 0.5$ $f'(0.5) = -2e^{-1} - 0.5$

$x_2 = 0.5 - \frac{e^{-1} - 0.25}{-2e^{-1} - 0.5}$ $x_2 = 0.595$

Limitations of the Newton-Raphson method

As the method uses the tangent to the curve, if the starting value is a stationary point or close to a stationary point (min, max or inflection) the method does not work

d) <u>APPROXIMATING THE AREA UNDER A CURVE</u>
TRAPEZIUM RULE – given in the formula book but make sure you know how to use it!

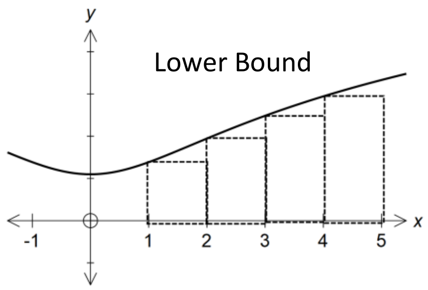
The trapezium rule gives an approximation of the area under a graph

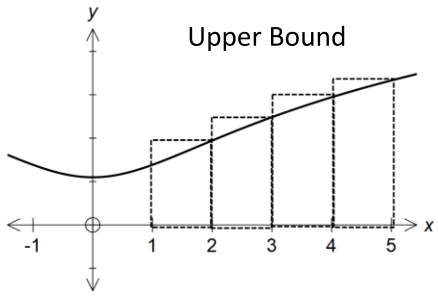
$$\int_a^b y\,dx \approx \frac{1}{2}h[(y_{0+}y_n) + 2(y_1 + y_2 + \ldots y_{n-1})] \quad \text{where } h = \frac{b-a}{n}$$

An easy way to calculate the y values is to use the TABLE function on a calculator – make sure you list the values in the formula (or a table) to show your method

- The rule will **underestimate** the area when the curve is **concave**
- The rule will **overestimate** the area when the curve is **convex**

UPPER and LOWER bounds - Area estimated using the area of rectangles
For the function shown below if the left hand 'heights' are used the total area is a Lower Bound – the rectangles calculated using the right hand heights the area results in the Upper Bound





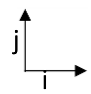
9 VECTORS

A vector has two properties **magnitude** (size) and **direction**

a) NOTATION

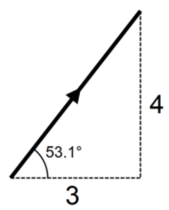
Vectors can be written as

a = $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

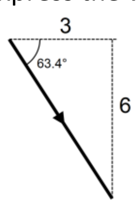


a = 3i + 4j where i and j perpendicular unit vectors (magnitude 1)

Magnitude-direction form (5, 53.1º) also known as **polar** form
The direction is the angle the vector makes with the **positive x axis**



Express the vector **p** = 3i – 6j in polar form



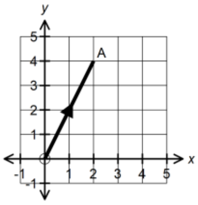
|**p**| = $\sqrt{3^2 + (-6)^2}$

= $3\sqrt{5}$

p = ($3\sqrt{5}$, 63.4º)

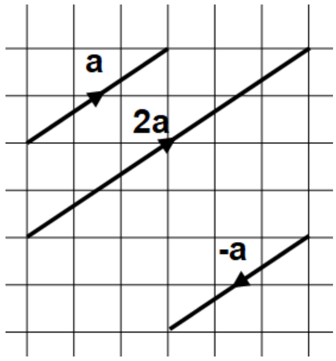
The **Magnitude** of vector **a** is denoted by |**a**| and can be found using Pythagoras |**a**| = $\sqrt{3^2 + 4^2}$
A **Unit Vector** is a vector which has magnitude 1

A **position vector** is a vector that starts at the origin (it has a fixed position)



$\overrightarrow{OA} = \binom{2}{4} \qquad 2i + 4j$

b) <u>ARITHMETIC WITH VECTORS</u>



Multiplying by a scalar (number)

a $= \binom{3}{2}$ 3i + 2j

2**a** $= 2\binom{3}{2} = \binom{6}{4}$ 6i + 4j

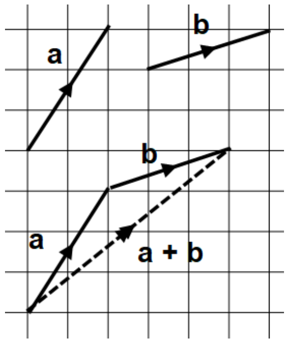
a and **2a** are **parallel** vectors

Multiplying by -1 reverses the direction of the

Addition of vectors

a $= \binom{2}{3}$ **b** $= \binom{3}{1}$

a + b $= \binom{2}{3} + \binom{3}{1} = \binom{5}{4}$

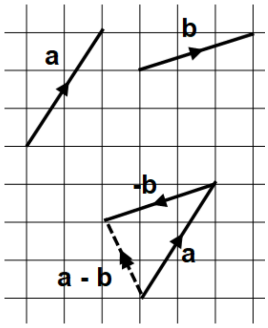


Subtraction of vectors

a $= \binom{2}{3}$ **b** $= \binom{3}{1}$

a - b $= \binom{2}{3} - \binom{3}{1} = \binom{-1}{2}$

This is really **a + -b**



A and B have the coordinates (1,5) and (-2,4).

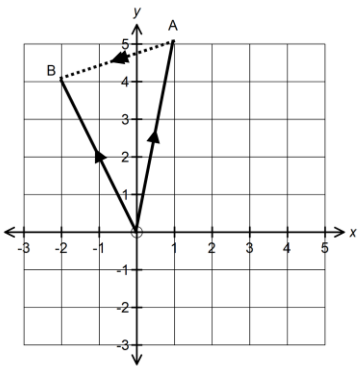
a) Write down the position vectors of A and B

$\overrightarrow{OA} = \binom{1}{5} \quad \overrightarrow{OB} = \binom{-2}{4}$

b) Write down the vector of the line segment joining A to B

$\overrightarrow{\boldsymbol{AB}} = -\overrightarrow{\boldsymbol{OA}} + \overrightarrow{\boldsymbol{OB}} \quad \boldsymbol{or} \quad \overrightarrow{\boldsymbol{OB}} - \overrightarrow{\boldsymbol{OA}}$

$\overrightarrow{AB} = \binom{-2}{4} - \binom{1}{5} = \binom{-3}{-1}$



Collinear - vectors in 2D and 3D can be used to show that 3 or more points are collinear (lie on a straight line)

Show that A(3,1,2) B(7,4,5) and C(19,13,14)

$\overrightarrow{AB} = 4i + 3j + 3k \qquad \overrightarrow{BC} = 12i + 9j + 9k$

$\overrightarrow{BC} = 3\overrightarrow{AB}$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors sharing a common point B and are therefore collinear

10 PROOF

Notation *If* $x = 3$ *then* $x^2 = 9$

$\Rightarrow$ $x = 3 \Rightarrow x^2 = 9$

$x = 3$ is a condition for $x^2 = 9$

$\Leftarrow$ $x = 3 \Leftarrow x^2 = 9$ is **not true** as x could = - 3

$\Leftrightarrow$ $x + 1 = 3 \Leftrightarrow x = 2$

a) Proof by deduction – statement proved using known mathematical principles

Useful expressions : 2n (an even number) 2n + 1 (an odd number)

Prove that the difference between the squares of any consecutive even numbers is a multiple of 4

Consecutive even numbers 2n, 2n + 2

$(2n + 2)^2 - (2n)^2$

$4n^2 + 8n + 4 - 4n^2$

$= 8n + 4$

$= 4(2n + 1)$ a multiple of 4

b) Proof by exhaustion – showing that a statement is true for every possible case or value

Prove that $(n + 2)^3 \geq 3^{n-1}$ for $n \epsilon \mathrm{N}$, n<4

We need to show it is true for 1,2 and 3

n = 1 $27 \geq 1$

n = 2 $64 \geq 3$

n = 3 $125 \geq 9$

True for all possible values hence proof that the statement is true by exhaustion

c) Disproof by counter example – finding an example that shows the statement is false.

Find a **counter example** for the statement '2n + 4 is a multiple of 4'

n = 2 4 + 4 = 8 a multiple of 4

n = 3 6 + 4 = 10 NOT a multiple of 4

d) Proof by contradiction - assume first that the statement is not true and then show that this is not possible

Prove that for all integers n, if $n^3 + 5$ is odd then n is even

Assume that $n^3 + 5$ is odd and n is odd

Let $n^3 + 5 = 2k + 1$ and let $n = 2m + 1$ (k and m integers)

$2k + 1 = (2m + 1)^3 + 5$

$2k + 1 = 8m^3 + 12m^2 + 6m + 6$

$2k = 8m^3 + 12m^2 + 6m + 5$

$2k = 2(4m^3 + 6m^2 + 3m) + 5$

$k = (4m^3 + 6m^2 + 3m) + \frac{5}{2}$

$4m^3 + 6m^2 + 3m$ has an integer value leaving k as a non-integer value (contradicting assumptions)

Our initial assumption that when that $n^3 + 5$ is odd then n is odd is false so n must be even