

## 1. Proof

### Proof by Induction

- Step 1 : Show that the rule works for the first value e.g.  $n = 1$
- Step 2 : Make a statement : *Assume that the result holds for  $n = k$*
- Step 3 : Show that the results holds for  $n = k + 1$

### Summation of a series

**Example** Prove by induction that  $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$

Step 1 : Check for  $n=1$

$$\text{LHS : } 1^2 = 1 \quad \text{RHS : } \frac{1}{6} \times 2 \times 3 = 1$$

*So the formula is true for  $n=1$*

Step 2 : Assume that the result holds for the  $k^{\text{th}}$  step, when  $n=k$  then  $\sum_{r=1}^k r^2 = \frac{k}{6}(k+1)(2k+1)$

Step 3 : Show that the results holds for  $k + 1$

$$\begin{aligned} \sum_{r=1}^{k+1} r^2 &= \sum_{r=1}^k r^2 + (k+1)^2 \quad \text{so} \quad \sum_{r=1}^{k+1} r^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2 \\ &= (k+1) \left( \left( \frac{k}{6} \right) (2k+1) + (k+1) \right) \\ &= \frac{(k+1)}{6} (2k^2 + 7k + 6) \\ &= \frac{(k+1)}{6} ((k+1) + 1)(2(k+1) + 1) \end{aligned}$$

*This matches the formula with  $n$  replaced by  $k+1$ .*

*Take out any common factors as soon as you can as you work – not a good idea to expand and simplify fully before looking for factors*

If the statement is true for  $n = k$  then it is also true for  $n = k+1$ .  
As the statement is true for  $n = 1$  it must be true for all  $n \geq 1$

### Multiples

**Example** Prove by induction that  $9^n - 1$  is a multiple of 8

Step 1 : Check for  $n=1$   $9^1 - 1 = 8$  which is a multiple of 8

Step 2 : Assume that the result holds for a value  $n = k$

Then  $9^k - 1 = 8m$  where  $m$  is an integer

Step 3 : Show that the results holds for  $k + 1$

$$\begin{aligned} 9^{k+1} - 1 &= 9^k \times 9 - 1 \quad \text{but} \quad 9^k = 8m + 1 \\ \text{so } 9^{k+1} - 1 &= (8m + 1) \times 9 - 1 \\ &= 72m + 9 - 1 \\ &= 8(9m + 1) \quad \text{so } 9^{k+1} - 1 \text{ is also a multiple of 8} \end{aligned}$$

If the statement is true for  $n = k$  then it is also true for  $n = k+1$ .  
As the statement is true for  $n = 1$  it must be true for all  $n \geq 1$

## 2. Complex Numbers

$$z = a + ib$$

real      imaginary

|                                 |
|---------------------------------|
| $i = \sqrt{-1} \qquad i^2 = -1$ |
|---------------------------------|

- Addition and Subtraction**

$$(2 + 3i) + (5 - 2i) = 7 + i \quad (\text{add/subtract real part then imaginary part})$$

- Multiplication** - multiply out the same way you would  $(x - 2)(x + 4)$

**EXAMPLE**  $(2 - 3i)(6 + 2i) = 12 + 4i - 18i - 6i^2$

$$= 12 - 14i + 6$$

$$= 18 - 14i$$

- Complex Conjugate  $z^*$**

If  $z = a + ib$  then its complex conjugate is  $z^* = a - ib$

*always collect the 'real' and 'imaginary' parts before looking for the conjugate*

- Solving Equations** - if two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

**EXAMPLE** Find  $z$  when  $5z - 2z^* = 3 - 14i$

Let  $z = x + iy$  and so  $z^* = x - iy$

$$5(x + iy) - 2(x - iy) = 3 - 14i$$

$$3x + 7iy = 3 - 14i$$

Equating real :  $3x = 3$  so  $x = 1$

Equating imaginary :  $7y = -14$  so  $y = -2$        $z = 1 - 2i$

- Quadratic equations –  $ax^2 + bx + c = 0$**  (where  $a, b$  and  $c$  are real)

If  $b^2 - 4ac < 0$  then the equation has 2 complex roots which will be **complex conjugates**

**EXAMPLE** Solve  $4x^2 + 2x + 1 = 0$      $a = 4$     $b = 2$     $c = 1$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 4 \times 1}}{2 \times 4}$$

$$x = \frac{-2 \pm \sqrt{-12}}{8} \qquad x = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$$

- Cubic equations –  $ax^3 + bx^2 + cx + d = 0$**

3 real roots

1 real root and 1 pair of complex conjugate roots

**EXAMPLE**

The cubic equation  $ax^3 + bx^2 + cx + d = 0$  has roots  $x = 2$  and  $x = 1 - 2i$  and a third root. Find the value of the constants  $a, b, c$  and  $d$

Third root is  $1 + 2i$  (complex conjugate)

$$(x - 2)(x - (1 - 2i))(x - (1 + 2i)) = ax^3 + bx^2 + cx + d$$

$$(x - 2)(x^2 - 2x + 5)$$

$$= x^3 - 2x^2 + 5x - 2x^2 + 4x - 10$$

$$= x^3 - 4x^2 + 9x + 10$$

$$a = 1, b = -4, c = 9, d = 10$$

$$(x - 1 + 2i)(x - 1 - 2i)$$

$$= x^2 - x - 2xi - x + 1 + 2i + 2xi - 2i + 4$$

$$= x^2 - 2x + 5$$

- Quartic equations –  $ax^4 + bx^3 + cx^2 + dx + e = 0$**

4 real roots

2 real roots and 1 pair of complex conjugate roots

2 pair of complex conjugate roots

- **ARGAND DIAGRAMS**

$$z = x + iy$$

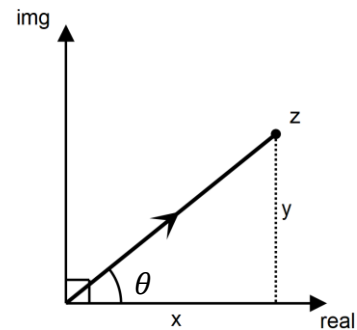
$$|z| = \sqrt{x^2 + y^2}$$

$$\arg(z) = \theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$\arg$  usually in radians where  $-\pi < \theta < \pi$

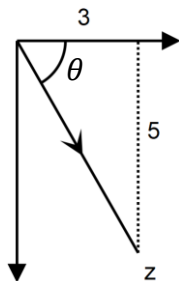
**anti-clockwise** from x axis gives a positive argument



- **Modulus-Argument form**  $z = r(\cos \theta + i \sin \theta)$  where  $r = |z|$  and  $\theta = \arg z$

**EXAMPLE**

Write the complex number  $z = 3 - 5i$  in modulus-argument form (radians to 2 d.p)



$$r = \sqrt{3^2 + 5^2}$$

$$= \sqrt{34}$$

$$\theta = \tan^{-1}\left(\frac{5}{3}\right)$$

$$\theta = -1.03$$

$$z = \sqrt{34}(\cos(-1.03) + i \sin(-1.03))$$

- **Multiplying and dividing complex numbers**

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

**Multiplication**

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

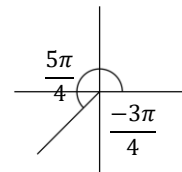
**Division**

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

You may need to adjust the arg to ensure that it lies within the range  $-\pi < \theta < \pi$

$$\text{If } \theta_1 + \theta_2 = \frac{5\pi}{4}$$

$$= \frac{-3\pi}{4}$$



- **LOC**

**Circles**

$|z| = a$  circle radius  $a$  centre at the origin

$|z - z_1| = a$  circle radius  $a$  centre at  $z_1$

$$|z - 5 - 3i| = 4$$

$$|z - (5 + 3i)| = 4 \quad \text{Centre } 5 + 3i \quad \text{radius } 4$$

**Half Lines** - straight lines which start at a specific point and only have one direction.

$\arg z = \theta$  represents the **half** line starting at 0 inclined at an angle  $\theta$  to the positive x direction

$\arg(z - z_1) = \theta$  represents the **half** line starting at  $z_1$  inclined at an angle  $\theta$  to the positive x direction

**EXAMPLE**

For  $\arg(z - 3 - 2i) = \frac{3\pi}{4}$  construct the locus of  $z$

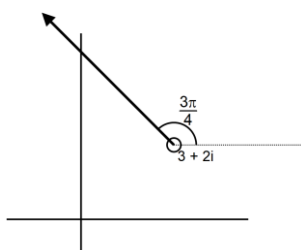
$$z = x + iy \quad \arg(z - (3 + 2i)) = \frac{3\pi}{4} \quad \arg(x + iy - (3 + 2i)) = \frac{3\pi}{4}$$

$$\tan^{-1}\left(\frac{y-2}{x-3}\right) = \frac{3\pi}{4}$$

$$\frac{y-2}{x-3} = -1$$

$$x + y = 5$$

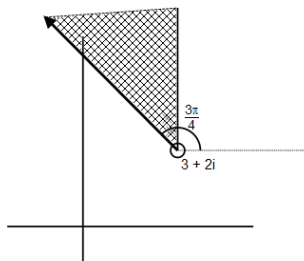
Restricted to  $x < 3$  and  $y > 2$



## Regions

If the complex locus is an inequality then it will describe a region

$$\frac{\pi}{2} \leq \arg(z - 3 - 2i) \leq \frac{3\pi}{4}$$



## Exponential form of complex numbers

**Euler's relation**  $e^{i\theta} = \cos\theta + i\sin\theta$   
 $-\pi < \theta < \pi$

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$-\pi < \theta < \pi$$

**EXAMPLE:** Express  $\sqrt{3}(\cos(\frac{\pi}{3}) - i\sin(\frac{\pi}{3}))$  in the form  $re^{i\theta}$

must be in the form  $r(\cos\theta + i\sin\theta)$

Using  $\cos(-\theta) = \cos\theta$   
 $\sin(-\theta) = -\sin\theta$

$$\sqrt{3}(\cos(\frac{\pi}{3}) - i\sin(\frac{\pi}{3})) = \sqrt{3}(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})) = \sqrt{3}e^{-i\frac{\pi}{3}}$$

**Multiplication and division** where  $z_1 = r_1e^{i\theta_1}$  and  $z_2 = r_2e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

## De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

For all integer values of n  
 (can easily be proved by induction)

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

**Take care with negative values to show method**

$$\begin{aligned} (\cos\theta - i\sin\theta)^n &= (\cos(-\theta) + i\sin(-\theta))^n \\ &= \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta - i\sin n\theta \end{aligned}$$

*Make sure you show all of this*

*DO NOT write*

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$$

Can be used to simplify complex calculations

### EXAMPLE

Calculate  $(1 + \sqrt{3})^6$  Let  $z = 1 + \sqrt{3}$   $|z| = \sqrt{3}$

$$\arg z = \frac{\pi}{3}$$

$$z = \sqrt{3}(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}))$$

$$\begin{aligned} z^6 &= 27 \left( \cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right) \right) \\ &= 27 \end{aligned}$$

## Multiple-angle trig functions to powers of trig functions

### EXAMPLE

Show that  $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \quad (\text{real parts})$$

$$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$

$$= 4\cos^3\theta - 3\cos\theta$$

### Powers of trig functions to multiple-angle trig functions

$$z^n = \cos n\theta + i \sin n\theta \quad z^{-n} = \cos(-n\theta) + i \sin(-n\theta) \\ = \cos n\theta - i \sin n\theta$$

$$\text{Leads to } \frac{z^n + z^{-n}}{2} = \cos n\theta \quad \frac{z^n - z^{-n}}{2i} = \sin n\theta$$

#### EXAMPLE

$$\text{Show that } \sin^4 \theta = \frac{\cos 4\theta - \cos 2\theta + 6}{8}$$

$$\text{Let } z = \cos \theta + i \sin \theta \quad 2i \sin \theta = z - z^{-1}$$

$$(2i \sin \theta)^4 = (2i)^4 \sin^4 \theta \\ = (z - z^{-1})^4 = z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4} \\ = (z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6 \\ (2i)^4 \sin^4 \theta = 2 \cos 4\theta - 8 \cos 2\theta + 6$$

$$\sin^4 \theta = \frac{\cos 4\theta - \cos 2\theta + 6}{8}$$

### Roots of Unity $z^n = 1$

The equation  $z^n = 1$  has roots  $\omega = e^{\frac{2k\pi i}{n}}$   $k = 0, 1, 2, \dots, (n-1)$

When plotted the roots are on a unit circle at intervals of  $\frac{2\pi}{n}$  radians

#### EXAMPLE

Find the cube roots of unity

$$\omega = e^{\frac{2k\pi i}{n}} \quad k=0 \quad \omega = e^0 = 1 \\ k=1 \quad \omega = e^{\frac{2\pi i}{3}} \quad \left( \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ k=2 \quad \omega = e^{\frac{4\pi i}{3}} \quad \left( \cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

- For the  $n$ th roots of unit ( $n > 1$ ) the sum of the roots is zero
- When plotted the roots are on a unit circle at intervals of  $\frac{2\pi}{n}$  radians

### Roots $z^n = \alpha$ has $n$ solutions

#### EXAMPLE

Find all the solutions to  $z^3 = 1 - i$

$$\text{Let } z = re^{i\theta} \quad \text{so } z^3 = r^3 e^{i3\theta} \quad 1 - i = \sqrt{2} e^{i\left(-\frac{\pi}{4} + 2\pi k\right)}$$

$$r^3 e^{i3\theta} = \sqrt{2} e^{i\left(-\frac{\pi}{4} + 2\pi k\right)} \quad r^3 = \sqrt{2} \quad r = 2^{\frac{1}{6}}$$

$$e^{i3\theta} = e^{i\left(-\frac{\pi}{4} + 2\pi k\right)} \quad 3\theta = -\frac{\pi}{4} + 2\pi k$$

$$\theta = \frac{8\pi k - 1}{12}$$

$$\text{Roots are } 2^{\frac{1}{6}} e^{i\left(\frac{8\pi k - 1}{12}\right)} \quad k = 0, 1 \text{ and } 2$$

### 3. Matrices

- **Order of a matrix** 'm x n' where m is the number of rows and n the number of columns  
(If order is correct for a particular operation then matrices are described as **conformable**)

**Zero Matrix** – all elements are 0

**Identity matrix**  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$   $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$   $AI = IA = I$

**Transpose** of a matrix  $A^T$  write the rows as columns and the columns as rows  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Special result  $(AB)^T = B^T A^T$

**Determinant** – every square matrix has a determinant  
- if determinant is zero the matrix is **singular**

**2 x 2**  $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$   $\det M = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$   $\det M$  is the area scale factor

**3 x 3**  $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$   $\det M = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$   
 $\det M$  is the volume scale factor

Adding a scalar multiple of a row/column to another row/column does not change the determinant of the matrix

**Factorising the determinant**

Works in same way as factor theorem if  $(x - a)$  is a factor of  $|M|$  then  $|M| = 0$  when  $x = a$

**EXAMPLE** : Factorise the determinant of  $\begin{vmatrix} a & b & c \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix}$

$$C1 = C1 - C2 \quad \begin{vmatrix} a-b & b & c \\ bc-ca & ca & ab \\ b-a & c+a & a+b \end{vmatrix} = (b-a) \begin{vmatrix} -1 & b & c \\ c & ca & ab \\ 1 & c+a & a+b \end{vmatrix}$$

$$C2 = C2 - C3 \quad (b-a) \begin{vmatrix} -1 & b-c & c \\ c & a(c-b) & ab \\ 1 & c-b & a+b \end{vmatrix} = (b-a)(c-b) \begin{vmatrix} -1 & -1 & c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix}$$

$$R1 = R1 + R3 \quad (b-a)(c-b) \begin{vmatrix} 0 & 0 & a+b+c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix} = (b-a)(c-b)(c-a)(a+b+c)$$

**Inverses**  $A \times A^{-1} = I$  (If a matrix is singular  $\det A = 0$  then it is not invertible (no inverse))

**2 x 2**  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  the inverse  $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

**3 x 3**  $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$  the inverse  $M^{-1} = \frac{1}{\det A} \begin{bmatrix} A & -D & G \\ -B & E & -H \\ C & -F & I \end{bmatrix}$

where  $A = \begin{vmatrix} e & f \\ g & h \end{vmatrix}$   $B = \begin{vmatrix} d & f \\ g & i \end{vmatrix}$   $C = \begin{vmatrix} d & e \\ g & h \end{vmatrix}$   $D = \begin{vmatrix} b & c \\ g & i \end{vmatrix}$   $E = \begin{vmatrix} a & c \\ g & i \end{vmatrix}$  ....

**Can be used to solve simultaneous equations**

**EXAMPLE** Solve  $x + 3y + 5z = -2$   $3x - 4y + 2z = 7$   $3x + 11y + 13z = 2$

$$\mathbf{AX} = \mathbf{B} \quad \begin{bmatrix} 1 & 3 & 5 \\ 3 & -4 & 2 \\ 3 & 11 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix} \quad \mathbf{A}^{-1} = \frac{1}{52} \begin{bmatrix} -74 & 16 & 26 \\ -33 & -2 & 13 \\ 45 & -2 & -13 \end{bmatrix}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{52} \begin{bmatrix} -74 & 16 & 26 \\ -33 & -2 & 13 \\ 45 & -2 & -13 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix} \quad x = 6 \quad y = 1.5 \quad z = -2.5 \quad \text{Unique solution}$$

Infinitely many solutions or no solution if  $\det \mathbf{A} = 0$

- Addition and Subtraction** – must have the same order

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 4 & 7 \end{bmatrix}$$

- Multiplication** – for  $\mathbf{AB}$  number of columns in  $\mathbf{A}$  must equal number of rows in  $\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 \times 2 + 2 \times 3 + 3 \times 4 & 1 \times -1 + 2 \times -2 + 3 \times 0 \\ 4 \times 2 + 5 \times 3 + 3 \times 4 & 4 \times -1 + 5 \times -2 + 6 \times 0 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -5 \\ 47 & -14 \end{bmatrix} \end{aligned}$$

- Transformations– ORDER MATTERS**

Make sure you know the exact trig ratios

To calculate the coordinates of a point after a transformation

**Multiply the Transformation Matrix by the coordinate**

Find the position of point (2,1) after a stretch of Scale factor 5 parallel to the x-axis

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix} \quad (10,1)$$

**To Identify a transformation from its matrix** (consider the unit square)

Consider the points (1,0) and (0,1)

$$\mathbf{M} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{point (1,0)} \quad \mathbf{M} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \times 1 + 0 \times 0 \\ 0 \times 1 + 2 \times 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad (1,0) \rightarrow (4,0)$$

$$\text{point (0,1)} \quad \mathbf{M} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \times 0 + 0 \times 1 \\ 0 \times 0 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad (0,1) \rightarrow (0,2)$$

Stretch parallel to the x axis scale factor 4 and stretch parallel to the y axis scale factor 2

Standard Transformations

## REFLECTIONS

$$\text{In the y axis } \mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{In the x-axis } \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{In the line } y = x \quad \mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{In the line } y = (\tan \theta)x \quad \mathbf{M} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

## ENLARGEMENT

Scale factor  $k$ , centre  $(0,0)$   $M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

## STRETCH

Scale factor  $a$  parallel to the  $x$ -axis and Scale factor  $b$  parallel to the  $y$ -axis  $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

## ROTATION

Centre  $(0,0)$  through  $\theta$  **anti-clockwise**  $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Centre  $(0,0)$  through  $\theta$  **clockwise**  $M = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

**ORDER MATTERS !!!!** A figure is transformed by  $M_1$  followed by  $M_2$   
Multiply  $M_2 M_1$

**WORKING IN 3D** -same as in 2D – consider the points  $(1,0,0)$   $(0,1,0)$  and  $(0,0,1)$

**Invariant Points** - Points which are unaffected by a transformation and remain fixed

**EXAMPLE** Find the invariant points of the transformation with matrix  $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad x + 2y = x \quad -y = y \quad (y = 0)$$

$y = 0$  the  $x + 0 = x$  so all points  $(x, 0)$  are invariant

## Equation of the line of invariant points

The transformation matrix  $A$  is given by  $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$  find the equation of the line of invariant points passing through the origin  $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad x' = 2x + 3y \quad y' = x + 4y \quad y' = mx' + c$

$$x + 4y = m(2x + 3y) + c$$

$$x + 4(mx + c) = m(2x + 3(mx + c)) + c$$

$$x(3m^2 - 2m - 1) + c(3m - 3) = 0$$

$$x(m - 1)(3m + 1) + c(3m - 3) = 0$$

If  $m = 1$   $c$  can take any value      If  $m = -\frac{1}{3}$   $c = 0$  (passing through the origin)  $y = -\frac{1}{3}x$

## Eigenvectors and Eigenvalues

An eigenvector is a non-zero vector representing a point on an invariant line after a linear transformation.

An eigenvalue is the scalar that maps the original point to the new point on the invariant line.

$M$  (matrix for a linear transformation)

$v$  (eigenvector)

$\lambda$  (eigenvalue)

$$Mv = \lambda v$$

Eigenvalue can be found by solving

$$\det(M - \lambda I) = 0 \quad (I \text{ is the identity matrix})$$

## Finding eigenvalues

### EXAMPLE

Given that 6 is an eigenvalue of the matrix  $M$  find the other two eigenvalues.

$$M = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 5 \\ 1 & 1 & -1 \end{bmatrix} \quad \det(M - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 5 \\ 1 & 1 & -1-\lambda \end{vmatrix}$$

$$6 \text{ is an eigenvalue so } 6 - \lambda \text{ is a factor} \quad = \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 2 & 3-\lambda & 5 \\ 1 & 1 & -1-\lambda \end{vmatrix}$$

$$= 6 - \lambda \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3-\lambda & 5 \\ 1 & 1 & -1-\lambda \end{vmatrix}$$

$$= 6 - \lambda \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3-\lambda & 5 \\ 0 & 0 & -2-\lambda \end{vmatrix}$$

$$= 6 - \lambda \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1-\lambda & 5 \\ 0 & 0 & -2-\lambda \end{vmatrix} \quad (6 - \lambda)((3 - \lambda)(-2 - \lambda)) - 2(-2 - \lambda) = 0$$

$$\begin{aligned} (6 - \lambda)(\lambda^2 + \lambda - 2) &= 0 \\ (6 - \lambda)(\lambda + 2)(\lambda - 1) &= 0 \\ \lambda &= 6, -2, 1 \end{aligned}$$

**Finding eigenvectors** find the eigenvector corresponding to eigenvalue=6 in the above example

### EXAMPLE

Find the eigenvector corresponding to eigenvalue=6 in the above example

$$\lambda = 6 \quad \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 5 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 6 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{aligned} 3x + 2y + 2z &= 6x \\ 2x + 3y + 5z &= 6y \\ x + y - z &= 6z \end{aligned}$$

$$-3x + 2y + 2z = 0 \quad 2x - 3y + 5z = 0 \quad x + y - 7z = 0$$

$$\textcircled{1} + 3\textcircled{3} \quad 5y = 19z \quad \text{subs into } \textcircled{2} \quad 2x - 3\left(\frac{19}{5}\right)z = 0 \quad 5x = 16z$$

$$y = \frac{19}{5}z \quad x = \frac{16}{5}z \quad \text{Using } z = 5 \quad \text{eigenvector} = \begin{bmatrix} 16 \\ 19 \\ 5 \end{bmatrix}$$

## Reducing a matrix to diagonal form (Diagonalisation)

constants in the top left and bottom right and zero's everywhere else)

If  $U$  is the matrix composed of eigenvectors of  $M$  as its columns and  $U$  is non-singular then

$D = U^{-1} M U$  where  $D$  is the matrix with the corresponding eigenvalues on its leading diagonal and zeros elsewhere.

$M = U D U^{-1}$  and  $M^n = U D^n U^{-1}$  (Diagonalisation – expressing in the form  $M = U D U^{-1}$ )

## 4. Polynomials

### Quadratics

For a quadratic equation  $ax^2 + bx + c = 0$  with roots  $\alpha$  and  $\beta$

$$\text{Sum of the roots } \alpha + \beta = -\frac{b}{a} \quad \text{Product of roots } \alpha\beta = \frac{c}{a}$$

Once the value  $\alpha + \beta$  and  $\alpha\beta$  have been found, new quadratic equations can be formed with roots :

| Roots                                    | Sum of roots                                        | Product of roots        |
|------------------------------------------|-----------------------------------------------------|-------------------------|
| $\alpha^2$ and $\beta^2$                 | $(\alpha + \beta)^2 - 2\alpha\beta$                 | $(\alpha\beta)^2$       |
| $\alpha^3$ and $\beta^3$                 | $(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ | $(\alpha\beta)^3$       |
| $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ | $\frac{\alpha + \beta}{\alpha\beta}$                | $\frac{1}{\alpha\beta}$ |

The new equation becomes  $x^2 - (\text{sum of new roots})x + (\text{product of new roots}) = 0$



Don't forget the "= 0"

### EXAMPLE

The roots of the quadratic equation  $3x^2 + 4x - 1 = 0$  are  $\alpha$  and  $\beta$

Determine a quadratic equation with integer coefficients which has roots  $\alpha^3\beta$  and  $\alpha\beta^3$

$$\text{Step 1 : } \alpha + \beta = -\frac{4}{3} \quad \text{and} \quad \alpha\beta = -\frac{1}{3}$$

$$\alpha + \beta = -\frac{4}{3} \quad \alpha\beta = -\frac{1}{3}$$

$$\begin{aligned} \text{Step 2 : Sum of new roots } \alpha^3\beta + \alpha\beta^3 &= \alpha\beta(\alpha^2 + \beta^2) \\ &= -\frac{1}{3}((\alpha + \beta)^2 - 2\alpha\beta) \\ &= -\frac{1}{3}\left(\left(-\frac{4}{3}\right)^2 - 2\left(-\frac{1}{3}\right)\right) \\ &= -\frac{1}{3}\left(\frac{16}{9} + \frac{2}{3}\right) \\ &= -\frac{22}{27} \end{aligned}$$

$$\begin{aligned} \text{Step 3 : Product of new roots } \alpha^3\beta\alpha\beta^3 &= (\alpha\beta)^4 \\ &= \frac{1}{81} \end{aligned}$$

$$\text{Step 4 : Form the new equation } x^2 + \frac{22}{27}x + \frac{1}{81} = 0 \quad 81x^2 + 66x + 1 = 0$$

**CUBIC EQUATIONS**  $ax^3 + bx^2 + cx + d = 0$  has roots  $\alpha, \beta$  and  $\gamma$

$$\alpha + \beta + \gamma = -\frac{b}{a} \quad \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} \quad \alpha\beta\gamma = -\frac{d}{a}$$

**QUARTIC EQUATIONS (and beyond....)**  $ax^4 + bx^3 + cx^2 + dx + e = 0$

$$\sum \alpha = -\frac{b}{a} \quad \sum \alpha\beta = \frac{c}{a} \quad \sum \alpha\beta\gamma = -\frac{d}{a} \quad \sum \alpha\beta\gamma\delta = \frac{e}{a}$$

### EXAMPLE

The quartic equation  $x^4 + 2x^3 + 14x + 15 = 0$  has one root equal to  $1 + 2i$ . Find the other three roots.

As the coefficients are real, it follows that  $1 - 2i$  is also a root (complex conjugate)

Hence  $(x - (1 - 2i))(x - (1 + 2i))$  is a quadratic factor of the quartic

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + ax + b)$$

Equating coefficients:  $2 = a - 2$

$$a = 4 \quad 15 = 5b$$

$$b = 3$$

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + 4x + 3)$$

the four roots are  $1 + 2i$ ,  $1 - 2i$ ,  $-3$ , and  $-1$ .

## 5. Series

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

### Key Points

$$\sum_{r=1}^n 1 = n$$

$$\sum 5r^2 = 5 \sum r^2$$

Always look carefully at the limits

$$\sum_{r=n+1}^{2n} r^3 = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^n r^3$$

### The method of differences

- Often seen as a 2 part question – i) show that ..... ii) hence find .....
- ALWAYS show clearly the cancelling process

Example

Express  $\frac{1}{r(r+2)}$  in partial fractions.

Hence show that  $\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$

$$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$$

$$A(r+2) + Br = 1$$

$$A = \frac{1}{2} \quad B = -\frac{1}{2}$$

$$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

$$\begin{aligned} \sum_{r=1}^n \frac{1}{r(r+2)} &= \frac{1}{2(1)} - \frac{1}{2(3)} + \frac{1}{2(2)} - \frac{1}{2(4)} + \frac{1}{2(3)} - \frac{1}{2(5)} + \frac{1}{2(4)} - \frac{1}{2(6)} + \dots + \frac{1}{2(n-1)} - \frac{1}{2(n+1)} + \frac{1}{2(n)} - \frac{1}{2(n+2)} \\ &= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \\ &= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)} \\ &= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)} \\ &= \frac{n(3n+5)}{4(n+1)(n+2)} \end{aligned}$$

## Maclaurin Series standard series given in the formula booklet

### Example

Find the Maclaurin expansion of  $f(x) = \ln(1 + 2x)$  up to the  $x^3$  term.

State the range of values for which it is valid

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\begin{aligned}\ln(1 + 2x) &= (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} \\ &= 2x - 2x^2 + \frac{8x^3}{3}\end{aligned}$$

$\ln(1 + x)$  valid for  $-1 < x < 1$

$\ln(1 + 2x)$  valid for  $-1 < 2x < 1$  so  $-\frac{1}{2} < x < \frac{1}{2}$

### Deriving a Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} \dots \dots \dots$$

#### Assumption

- $f(x)$  can be expressed as  $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \dots \dots$
- The series can be differentiated term by term
- The function  $f(x)$  and all of its derivatives exist at  $x = 0$

If you are finding products of expansions check the range of values for which each is valid

### Limits of series

An **indeterminate form** is when given two functions  $f(x)$  and  $g(x)$  both approaching to 0 as  $x$  approaches some Limit, the limit cannot be found just by using the limits of  $f(x)$  and  $g(x)$  – usually division by 0 or involves infinity

### Finding limits using the Maclaurin series

Find the series in powers of  $x$  and investigate what happens as  $x \rightarrow 0$

### Finding limits using L'Hopitals Rule

If  $\lim_{x \rightarrow c} f(x)$  and  $\lim_{x \rightarrow c} g(x)$  are both 0 or  $\infty$  then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$  May need to differentiate again

#### EXAMPLE

$$\text{Find } \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} \quad \text{Let } f(x) = x - 2 \quad g(x) = x^2 - 4 \quad \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} g(x) = 0$$

$$\lim_{x \rightarrow 2} \frac{f'(x)}{g'(x)} = \frac{1}{2(2)} = \frac{1}{4}$$

## 6. Inequalities

### Polynomials

Rearrange to the form  $f(x) < > 0$

Identify the roots (critical values)  $f(x) = 0$

Sketch the graph to identify the ranges of  $x$  for which the inequality is satisfied

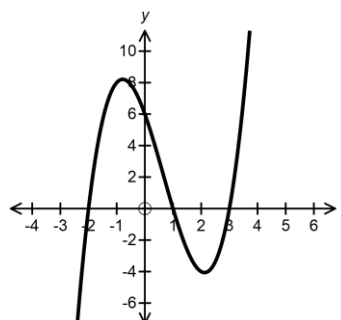
#### EXAMPLE

$$\text{Solve } x^3 - 2x^2 - 5x + 6 > 0$$

$$x^3 - 2x^2 - 5x + 6 = 0$$

$$(x - 1)(x - 3)(x + 2) = 0$$

$$\{x \in \mathbb{R}: -2 < x < 1\} \cup \{x \in \mathbb{R}: x > 3\}$$

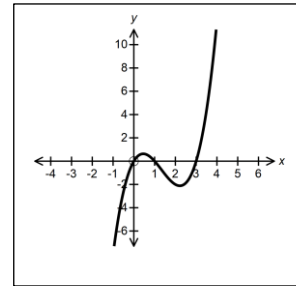


## Algebraic Fractions

**Multiply** through by the square of each denominator and solve as above

### EXAMPLE

$$\begin{aligned}\text{Solve } \frac{2x}{x-1} &> x & 2x(x-1) &> x(x-1)^2 \\ & & 2x^2 - 2x &> x^3 - 2x^2 + x \\ & & x^3 - 4x^2 + 3x &< 0 \\ & & x(x-3)(x-1) &< 0 \\ & & \{x \in \mathbb{R} : x < 0\} \cup \{x \in \mathbb{R} : 1 < x < 3\}\end{aligned}$$



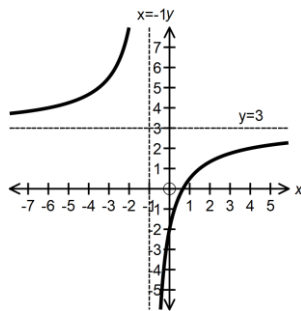
## 7. Graphs

### Rational Functions

**Linear denominator and numerator** :  $y = \frac{ax+b}{bx+c}$  1 vertical asymptote and 1 horizontal asymptote

Example

Sketch the graph of  $y = \frac{3x-2}{x+1}$



**Vertical Asymptote** :  $x + 1 = 0 \quad x = -1$

**Horizontal Asymptote** :

divide all terms by the highest power of x

$$\frac{3 - \frac{2}{x}}{1 + \frac{1}{x}} \quad \text{as } x \rightarrow \infty \quad \frac{3 - \frac{2}{x}}{1 + \frac{1}{x}} \rightarrow 3 \quad y = 3$$

Check axis intercepts  $x = 0 \quad y = -2 \quad y = 0 \quad x = \frac{2}{3}$

### Quadratic denominator and/or numerator

Find the asymptotes as above but the graph may cross the horizontal asymptote

If the denominator has no real roots then there is no vertical asymptote

A graph may have an **oblique asymptote** instead of a horizontal asymptote (see below)

### EXAMPLE

Sketch the graph of  $y = \frac{x^2+4x+4}{x+1}$

$x = 0 \quad y = 4 \quad y = 0 \quad x = -2$

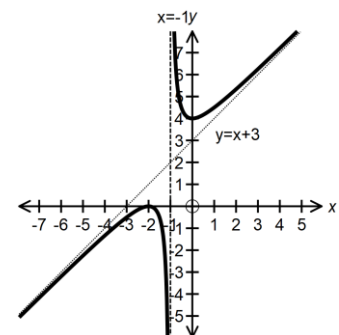
Vertical asymptote  $x = -1$

Dividing through by  $x^2$  results with an expression that is undefined as  $x \rightarrow \infty$

Instead work out  $(x^2 + 4x + 4) \div (x + 1)$

which gives  $x + 3$  (and a remainder of 1)

The equation of the **oblique asymptote** is  $y = x + 3$



### Stationary points – using the discriminant

To find stationary points set  $f(x) = k$  and rearrange to form an equation  $ax^2 + bx + c = 0$

Calculate the discriminant

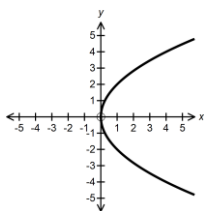
$b^2 - 4ac > 0$  the line  $y = k$  intersects the graph twice

$b^2 - 4ac < 0$  the line  $y = k$  goes not intersect the graph

$b^2 - 4ac = 0$  the line touches the graph at a stationary point where  $y = k$

## Conic Sections

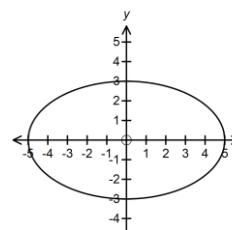
**Parabolas**  $y^2 = 4ax$



**Ellipses**  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

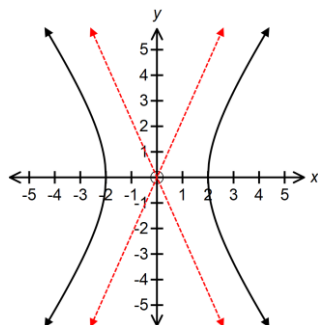
$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

$a > b$   
major axis =  $2a$   
minor axis =  $2b$



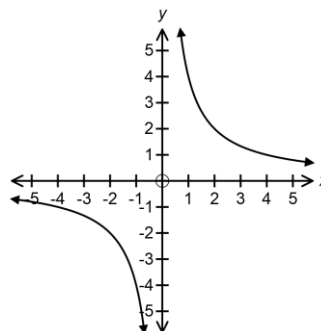
**Hyperbolas**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{asymptotes } y = \pm \frac{b}{a}x$$



**Rectangular Hyperbola (hyperbola where  $a = b$ )**

$$xy = c^2$$



## Transformations

**Translation**  $\begin{bmatrix} a \\ b \end{bmatrix}$

Replace  $x$  with  $(x - a)$

Replace  $y$  with  $(y - b)$

**Reflection in the line  $y = x$**

Replace  $x$  with  $y$  and vice versa

**Stretch Parallel to the x-axis scale factor  $a$**  Replace  $x$  with  $\frac{x}{a}$

**Stretch Parallel to the y-axis scale factor  $b$**  Replace  $y$  with  $\frac{y}{b}$

## 8. Integration

### Volumes of revolution

Rotation about the  $x$  - axis  $\int_{x=a}^{x=b} \pi y^2 dx$

Rotation about the  $y$  - axis  $\int_{y=a}^{y=b} \pi x^2 dy$

**Don't forget the  $\pi$  in your answer**

### Mean value of a Function

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$$

**Surface Area of Revolution** – formulae given in the formula booklet

Surface Area when rotated about the  $x$ -axis  $S_x = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

May be given in parametric form  $S_x = 2\pi \int_{t_1}^{t_2} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

**Arc length of a curve** - formulae given in formula booklet

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{Parametric form } S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### Improper Integrals

An integral is improper if

- One or both of the limits is infinity
- the integrand is undefined at one of the limits or somewhere in between the limits

Replace the 'undefined' limit with n and allow n to approach the limit once the function has been integrated

#### EXAMPLE

$$\text{Determine } \int_1^\infty \frac{1}{x^2} dx \quad \int_1^n \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^n \\ = -\frac{1}{n} + 1$$

As  $n \rightarrow \infty$  the value of the integral  $\rightarrow 1$

If an undefined value between the limits

split the integral e.g.  $\int_{-2}^2 \frac{1}{x^2} dx$

$$\int_{-2}^2 \frac{1}{x^2} dx = \int_{-2}^0 \frac{1}{x^2} dx + \int_0^2 \frac{1}{x^2} dx$$

Known Results – learn these

$$\lim_{x \rightarrow \infty} x^n e^{-x} = 0$$

$$\lim_{x \rightarrow 0^+} x^n \ln x = 0$$

**Integration with partial fractions** - learn the different forms

$$\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$$\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-a)^2}$$

$$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

**2 special forms** to look for (both given in the formula booklet)

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \quad |x| < a$$

**Deriving and using Reduction Formulae –**

Find a reduction formula to integrate  $\sin^n x$  with respect to x

$$\int \sin^n x dx = \int \sin^{n-1} x \sin x dx \quad (\text{integration by parts with } \frac{dv}{dx} = \sin x \text{ and } u = \sin^{n-1} x) \\ = -\sin^{n-1} x \cos x + \int \cos^2 x (n-1) \sin^{n-2} x dx \\ = -\sin^{n-1} x \cos x + \int (1 - \sin^2 x) (n-1) \sin^{n-2} x dx \\ = -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x - \int (n-1) \sin^n x dx$$

$$\int \sin^n x dx = -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x dx - \int (n-1) \sin^n x dx$$

$$\int \sin^n x dx + \int (n-1) \sin^n x dx = -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x dx$$

$$\int n \sin^n x dx = -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x dx$$

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \int \frac{1}{n} (n-1) \sin^{n-2} x dx$$

$$I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2} \quad \text{where} \quad I_1 = \int \sin x dx$$

**EXAMPLE**Evaluate  $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$ 

$$n=3 \quad I_3 = -\frac{1}{3} \sin^{3-1} x \cos x + \frac{3-1}{3} I_{3-2}$$

$$\begin{aligned} &= -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx \\ &= \left( -\frac{1}{3} \sin^2 \left( \frac{\pi}{2} \right) \cos \left( \frac{\pi}{2} \right) - \frac{2}{3} \cos \left( \frac{\pi}{2} \right) \right) - \left( 0 - \frac{2}{3} \right) \\ &= 0 + \frac{2}{3} \\ &= \frac{2}{3} \end{aligned}$$

## 9. Vectors (3D) $xi + yj + zk \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

**Magnitude**  $|xi + yj + zk| = \sqrt{x^2 + y^2 + z^2}$

**Straight lines**    **Vector Form**  $r = a + \lambda b$     **a**: position vector    **b**: direction vector (parallel to the line)

**Cartesian form** :  $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$     (x,y,z) is a general point on the line

$$r = a + \lambda b \quad \begin{aligned} a &= a_1i + a_2j + a_3k \\ b &= b_1i + b_2j + b_3k \end{aligned}$$

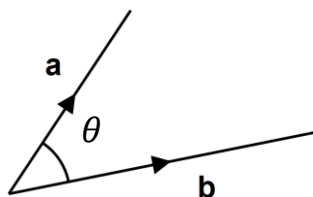
**EXAMPLE**

Find the vector equation of the line with cartesian equation  $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-5}{4}$

$$r = (3i - j + 5k) + \lambda(2i + 3j + 4k)$$

**Pairs of lines can** - be parallel (look at direction vector)  
 - intersect - a solution exists when solving simultaneously for all 3 variables  
 - be skew (never meet) - no solution when solving simultaneously

**Scalar product**  $a \cdot b = |a||b|\cos\theta$     where  $\theta$  is the angle between vectors a and b when both directed away from the same point  $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$



**Perpendicular vectors have scalar product of zero**

**Finding the angle between 2 lines**  $\cos\theta = \frac{a \cdot b}{|a||b|}$

**EXAMPLE:** Find the acute angle between the lines  $r = \begin{bmatrix} 0 \\ 7 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$  and  $r = \begin{bmatrix} -7 \\ 1 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$

Using the direction vectors

Let  $a = 4i - 3j + k$  and  $b = -3i + 2k$

$$a \cdot b = 4 \times -3 - 3 \times 0 + 1 \times 2 = -10 \quad |a| = \sqrt{26} \quad |b| = \sqrt{13}$$

$$\theta = \cos^{-1} \left( \frac{-10}{\sqrt{26}\sqrt{13}} \right) = 123^\circ \quad \text{Acute angle} = 57^\circ$$

## Perpendicular distance from a point to a line - shortest path

### EXAMPLE

Relative to a fixed origin O, the point A has position vector  $(8\mathbf{i} + 13\mathbf{j} - 2\mathbf{k})$ , the point B has position vector  $(10\mathbf{i} + 14\mathbf{j} - 4\mathbf{k})$ , and the point C has position vector  $(9\mathbf{i} + 9\mathbf{j} + 6\mathbf{k})$ .

Find the shortest distance from the point C to the line passing through A and B

$$\begin{aligned}\text{General point on line AB (point D)} &= (8\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \\ &= (8 + 2\lambda)\mathbf{i} + (13 + \lambda)\mathbf{j} + (-2 - 2\lambda)\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{CD} &= (9 - (8 + 2\lambda))\mathbf{i} + (9 - (13 + \lambda))\mathbf{j} + (6 - (-2 - 2\lambda))\mathbf{k} \\ &= (1 - 2\lambda)\mathbf{i} + (-4 - \lambda)\mathbf{j} + (8 + 2\lambda)\mathbf{k}\end{aligned}$$

$$\text{Perpendicular if } (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (1 - 2\lambda)\mathbf{i} + (-4 - \lambda)\mathbf{j} + (8 + 2\lambda)\mathbf{k} = 0$$

$$\begin{aligned}2(1 - 2\lambda) + 1(-4 - \lambda) - 2(8 + 2\lambda) &= 0 \\ 2 - 4\lambda - 4 - \lambda - 16 - 4\lambda &= 0 \\ -18 - 9\lambda &= 0 \quad \lambda = -2\end{aligned}$$

$$\text{When } \lambda = -2 \quad \mathbf{D} = 4\mathbf{i} + 11\mathbf{j} + 2\mathbf{k} \quad \mathbf{C} = 9\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}$$

$$\begin{aligned}\text{Distance between C and D} &= \sqrt{(9 - 4)^2 + (9 - 11)^2 + (6 - 2)^2} \\ &= 3\sqrt{5}\end{aligned}$$

**Perpendicular distance between parallel lines** - pick a fixed point on one of the lines and complete as in the above example

### Shortest distance between 2 skew lines

$$\mathbf{r}_1 = \mathbf{a} + \lambda\mathbf{b} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{c} + \mu\mathbf{d} \quad \text{Shortest distance} = \frac{|(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|}$$

## Planes

**Vector Form** :  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$   $\mathbf{a}$  is a fixed position vector

$\mathbf{b}$  and  $\mathbf{c}$  are non-parallel vectors in the plane  $\lambda$  and  $\mu$  scalars

### EXAMPLE

Find the vector equation of the plane that passes through points A(1,1,2) B(2, 0,3) and C(0,-2,1)

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$$

**Scalar Form**  $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = p$

$\mathbf{r}$  position vector of a general point on the plane

$\mathbf{a}$  position vector of a fixed point on the plane

$\mathbf{n}$  normal to the plane

$p$  a scalar constant

**Cartesian form**

$$\mathbf{r} \cdot \mathbf{n} = p \quad ax + by + cz + d = 0$$

$$p = -d$$

### EXAMPLE

Find the equation of the plane with normal vector  $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$  that contains point (3, 0, -2)

$$\mathbf{r} \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = 3 + 0 + 4 = 7 \quad x + 2y - 2z = 7$$

### Angle between a line and a plane

Plane  $\mathbf{r} \cdot \mathbf{n} = p$  and line  $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$      $\sin \phi = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}| |\mathbf{b}|}$

### Angle between two planes

$\mathbf{n}_1$  is the normal to plane A     $\mathbf{n}_2$  is the normal to plane B

The acute angle between the normals = the acute angle between the planes

Angle between the two planes is  $180 - \theta$  where  $\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|}$

#### EXAMPLE

Calculate the obtuse angle between the planes

$$\mathbf{r} \cdot (13\mathbf{i} - 9\mathbf{j} + 5\mathbf{k}) = 8$$

$$\mathbf{r} \cdot (2\mathbf{j} + 3\mathbf{k}) = 4$$

$$\mathbf{n}_1 = 13\mathbf{i} - 9\mathbf{j} + 5\mathbf{k} \quad \mathbf{n}_2 = 2\mathbf{j} + 3\mathbf{k}$$

$$\cos \theta = \frac{|(13\mathbf{i} - 9\mathbf{j} + 5\mathbf{k}) \cdot (2\mathbf{j} + 3\mathbf{k})|}{\sqrt{13^2 + (-9)^2 + 5^2} \sqrt{2^2 + 3^2}} = \frac{|-3|}{\sqrt{275} \sqrt{13}}$$

$$\theta = 87.1^\circ \text{ so the obtuse angle is } 92.9^\circ$$

### Point of intersection of a line and a plane

#### EXAMPLE

Find the point of intersection of the line  $\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$  and the plane  $\mathbf{r} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 13$

$$\text{Line } \mathbf{r} = \begin{bmatrix} 1 + 4\lambda \\ 2 + \lambda \\ 1 - 3\lambda \end{bmatrix} \quad \begin{bmatrix} 1 + 4\lambda \\ 2 + \lambda \\ 1 - 3\lambda \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 13$$

$$3 + 12\lambda - 2 - \lambda + 2 - 6\lambda = 13$$

$$3 + 5\lambda = 13$$

$$\lambda = 2$$

$$\text{Point of intersection} = (9, 4, -5)$$

### Intersecting Planes

Any two (non parallel planes) meet along a line – **line of intersection**

#### EXAMPLE

Find a cartesian equation of the line of intersection of the two planes  $\mathbf{r} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 11$  and  $\mathbf{r} \cdot \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = -7$

$$x - 2y + 3z = 11 \quad 2x + 2y - 2z = -7$$

$$(1) + (2) \quad 3x + z = 4 \quad z = 4 - 3x \quad x = \frac{4-z}{3}$$

$$\text{Subs into (2)} \quad 2x + 2y - 2(4 - 3x) = -7$$

$$8x + 2y = 1 \quad x = \frac{1-2y}{8}$$

$$x = \frac{1-2y}{8} = \frac{4-z}{3}$$

### 3 non-parallel planes can

- intersect at a single point
- all meet at the same line
- meet along different lines

### Perpendicular distance between a point and a plane

Point  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and plane  $\mathbf{r} \cdot \mathbf{n} = p$     Perpendicular distance =  $\frac{|\mathbf{a} \cdot \mathbf{n} - p|}{|\mathbf{n}|}$

## Vector Product

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta \hat{\mathbf{n}}$   $\theta$  is the angle between the 2 vectors when they are directed away from the same point

$\hat{\mathbf{n}}$  is a unit vector perpendicular to  $\mathbf{a}$  and  $\mathbf{b}$  (in a right-handed sense)

If 2 vectors are parallel then  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

The vector product is the determinant of a  $3 \times 3$  matrix

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Vector product to find the area of a triangle (based on  $\frac{1}{2}ab\sin C$ )

When  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$  pointing away from a point the area of the triangle formed is

$$\text{Area} = \frac{1}{2} |\mathbf{A} \times \mathbf{B}|$$

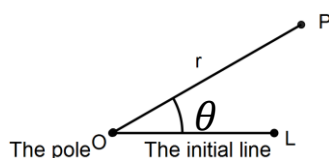
Example

Find the area of the triangle formed by  $\mathbf{a} = 2\mathbf{i} + \mathbf{k}$  and  $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 2 & -6 & 4 \end{vmatrix} \quad |(0-6)\mathbf{i} - (8-2)\mathbf{j} + (-12)\mathbf{k}| = \sqrt{(-6)^2 + (6)^2 + (-12)^2}$$
$$= \sqrt{216}$$

$$\text{Area} = \frac{\sqrt{216}}{2} = 3\sqrt{6}$$

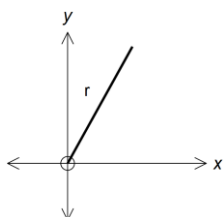
## 10. Polar Coordinates



P has polar coordinate  $(r, \theta)$



Polar – Cartesian



$$x = r\cos\theta \quad y = r\sin\theta$$

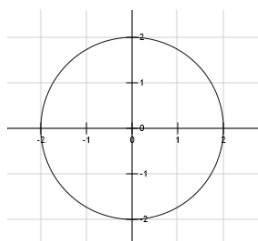
$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

### Plotting Curves and Graphs

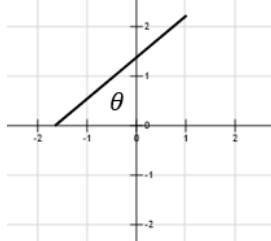
$$r = a$$

circle centre (0,0)  
radius a

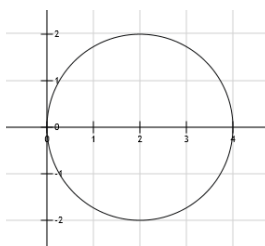


$$\theta = \alpha$$

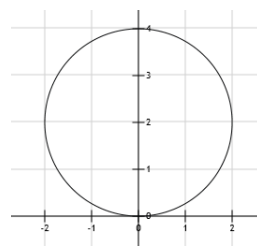
Straight line (semi-infinite/half line)



$$r = 2a \cos \theta$$



$$r = 2a \sin \theta$$



### Hints for sketching curves

- if  $r$  is expressed as a function of  $\cos$  only - **symmetrical about the initial line**
- if  $r$  is expressed as a function of  $\sin$  only - **symmetrical about the line  $\theta = \frac{\pi}{2}$**
- if  $r \rightarrow 0$  as  $\theta \rightarrow \infty$  then the line  $\theta = \alpha$  is a tangent to the curve at the pole
- **NEGATIVE VALUES** of  $r$  are not allowed – if  $r < 0$  in the interval  $\alpha < \theta < \beta$  then there is no curve in this interval

### Integrating Polar Curves

$$\int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

$r$  must be non-negative and defined for  $\alpha < \theta < \beta$

### MAKE SURE YOU CAN INTEGRATE TRIG FUNCTIONS

You may need to use

$$\int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta$$

$$\int \sin^2 \theta = \int \frac{1}{2} (1 - \cos 2\theta) = \frac{1}{2} (\theta - \frac{1}{2} \sin 2\theta)$$

$$\int \sin^2 2\theta d\theta = \int \frac{1}{2} (1 - \cos 4\theta) d\theta = \frac{1}{2} (\theta - \frac{1}{4} \sin 4\theta)$$

$$\int \sin^n \theta \cos \theta d\theta = \frac{1}{n+1} \sin^{n+1} \theta$$

## 11. Hyperbolic Functions

You need to learn these as they are not given in the formula booklet

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

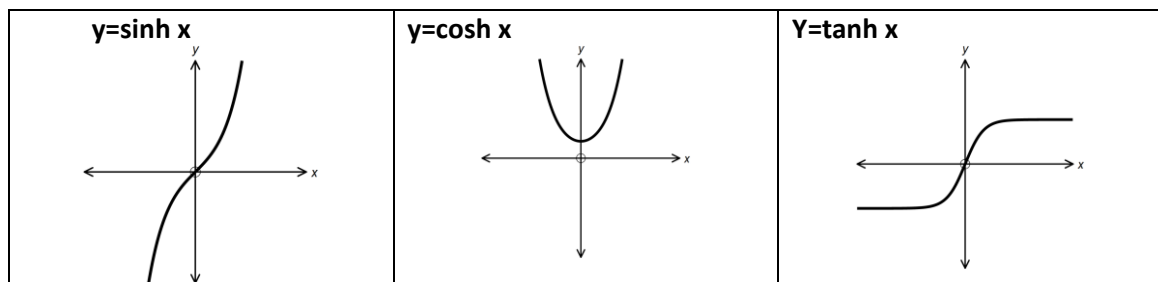
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

**EXAMPLE** Find the exact value of  $\cosh(\ln 4)$

$$\begin{aligned} \cosh(\ln 4) &= \frac{1}{2}(e^{\ln 4} + e^{-(\ln 4)}) \\ &= \frac{1}{2}\left(4 + \frac{1}{4}\right) = \frac{17}{8} \end{aligned}$$

**Identity:**  $\cosh^2 x - \sinh^2 x = 1$

### Graphs of hyperbolic functions



**Inverse hyperbolic functions – given in the formula booklet**

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\begin{aligned} \cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}) \\ x &\geq 1 \end{aligned}$$

$$\begin{aligned} \tanh^{-1} x &= \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \\ |x| &< 1 \end{aligned}$$

**EXAMPLE:** Solve  $\operatorname{artanh} x = \ln 4$

$$\frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) = \ln 4 \quad \ln \left( \frac{1+x}{1-x} \right) = \ln 16$$

$$\frac{1+x}{1-x} = 16$$

$$1 + x = 16 - 16x$$

$$x = \frac{15}{17}$$

## Reciprocal hyperbolic functions Use exponential forms to solve equations

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}} \quad \operatorname{sech} x = \frac{2}{e^x + e^{-x}} \quad \operatorname{coth} x = \frac{e^{2x} + 1}{e^{2x} - 1}$$

$$\text{Identities: } \operatorname{coth}^2 x - 1 = \operatorname{cosech}^2 x$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

### EXAMPLE

$$\text{Solve } 2\operatorname{coth}^2 x - 5\operatorname{cosech} x = 5$$

(answers in exact form)

$$\text{Using } \operatorname{coth}^2 x = \operatorname{cosech} x + 1$$

$$2(\operatorname{cosech} x + 1) - 5\operatorname{cosech} x = 5$$

$$2\operatorname{cosech} x + 2 - 5\operatorname{cosech} x - 5 = 0$$

$$2\operatorname{cosech} x - 5\operatorname{cosech} x - 3 = 0$$

$$(2\operatorname{cosech} x + 1)(\operatorname{cosech} x - 3) = 0$$

$$\operatorname{cosech} x = -\frac{1}{2} \quad \operatorname{cosech} x = 3$$

$$\sinh x = -2 \quad x = \sinh^{-1}(-2)$$

$$= \ln(-2 + \sqrt{(-2)^2 + 1})$$

$$= \ln(-2 + \sqrt{5})$$

$$\sinh x = \frac{1}{3} \quad x = \sinh^{-1}\left(\frac{1}{3}\right)$$

$$= \ln\left(\frac{1}{3} + \sqrt{\left(\frac{1}{3}\right)^2 + 1}\right)$$

$$= \ln\left(\frac{1}{3} + \sqrt{\frac{10}{9}}\right)$$

Calculus with hyperbolic functions sometimes easier to integrate or differentiate in exponential form

You need to learn these  $\frac{d}{dx} \sinh x = \cosh x$   $\frac{d}{dx} \cosh x = \sinh x$  (tan given in formula book)

$$\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \operatorname{coth} x \quad \frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x \quad \frac{d}{dx} \operatorname{coth} x = -\operatorname{cosec}^2 \theta$$

Derivatives of the inverse functions are given in the formula booklet

Look out for some of the standard integrals which can be used with a little substitution

### EXAMPLE

$$\text{Find } \int \frac{1}{\sqrt{x^2 + 14x + 24}} dx$$

$$= \int \frac{1}{\sqrt{(x+7)^2 - 25}} dx$$

$$\text{Let } x + 7 = 5\cosh u \quad u = \cosh^{-1}\left(\frac{x+7}{5}\right) \quad \frac{dx}{du} = 5\sinh u$$

$$\int \frac{1}{\sqrt{(5\cosh u)^2 - 25}} \times 5\sinh u du$$

$$= \int \frac{5\sinh u}{\sqrt{25\cosh^2 u - 25}} du$$

$$= \int \frac{5\sinh u}{5\sqrt{\cosh^2 u - 1}} du = \int \frac{5\sinh u}{5\sqrt{\sinh^2 u}} du = \int 1 du = u + c$$

$$\int \frac{1}{\sqrt{x^2 + 14x + 24}} dx = \cosh^{-1}\left(\frac{x+7}{5}\right) + c$$

## 12. Differential Equations

### First order differential equations

**Linear** 1<sup>st</sup> order differential equation  $\frac{dy}{dx} - \frac{y}{x} = 2x^2$  (dependent variable 'y' is linear)

A **General solution** has unknown variables and represents a family of solutions e.g.  $y = Ae^{x^2}$

To find a **Particular solution** boundary or initial conditions are used to evaluate unknown variables

e.g. when  $y = 2$   $x = 0$  or  $y(0) = 2$  both mean the same thing

### METHOD 1 : Separating the

$$\frac{dy}{dx} = 3x^2y$$

$$\frac{1}{y} dy = 3x^2 dx$$

$$\int \frac{1}{y} dy = \int 3x^2 dx$$

$$\ln|y| = x^3 + c$$

$$y = e^{x^3+c}$$

$$y = Ae^{x^3}$$

### METHOD 2 : Using an Integrating Factor

**Step 1 : Check that the equation is in the form**  $\frac{dy}{dx} + P(x)y = Q(x)$

$$x \frac{dy}{dx} - 2y = x^3 \quad \frac{dy}{dx} - \frac{2y}{x} = x^2$$

$$P(x) = -\frac{2}{x} \quad Q(x) = x^2$$

**Step 2 : Find the 'Integrating Factor'  $I(x) = e^{\int P(x)dx}$**

$$\int P(x)dx = \int \frac{-2}{x} dx = -2 \ln|x|$$

$$I(x) = e^{-2 \ln|x|} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

The left hand side is the result of differentiating  $I(x)y$  using the product rule

**Step 3 : Multiply all terms by the Integrating Factor**

$$\frac{1}{x^2} \frac{dy}{dx} - \frac{2y}{x^3} = 1$$

$$\frac{d}{dx} \left( \frac{y}{x^2} \right) = 1$$

$$\frac{y}{x^2} = \int 1 dx$$

$$\frac{y}{x^2} = x + c$$

$$y = x^3 + cx^2$$

$$\frac{d}{dx} (I(x)y) = I(x)Q(x)$$

MAKE SURE YOU SHOW THIS CLEARLY

Don't forget to include the 'c' in any rearrangement to isolate y  
e.g.  $cx^2$

**Step 4 : You have found the GENERAL solution – if necessary use values given to find a PARTICULAR solution**

## Second order differential equations

**Solving equations of the form**  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$

**Step 1 : Solve the auxiliary equation  $ak^2 + bk + c = 0$**

Solve

Find the general solution to  $2 \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} + 5y = 0$

$$2k^2 + 6k + 5 = 0 \quad (\text{Auxiliary equation})$$

$$k = \frac{-6 \pm \sqrt{6^2 - 4 \times 2 \times 5}}{2 \times 2}$$

$$k = -\frac{3}{2} \pm \frac{1}{2}i$$

3 possibilities for values for

|                                  |                         |                                          |
|----------------------------------|-------------------------|------------------------------------------|
| $k_1$ and $k_2$ real and unequal | $k_1 = k_2$ equal roots | $K_1 = p+qi \quad K_2 = p-qi$ (non real) |
| $y = Ae^{k_1x} + Be^{k_2x}$      | $y = e^{k_1x}(A + Bx)$  | $y = e^{px}(A \cos qx + B \sin qx)$      |

Exceptional case if solving  $\frac{d^2y}{dx^2} + n^2y = 0 \quad y = A \cos nx + B \sin nx$

## Step 2 : Use the chosen form of y for the general solution

General solution

$$y = e^{-\frac{3}{2}x} \left( A \cos \frac{1}{2}x + B \sin \frac{1}{2}x \right)$$

Solving equations of the form  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$

### Step 1 & 2 : Solve the auxiliary equation $ak^2 + bk + c = 0$ (as above)

This will find the COMPLEMENTARY Function , CF,  $y_c$

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 2y = 2e^{-2x}$$

Auxiliary equation

$$\begin{aligned} k^2 + 3k + 2 &= 0 \\ (k + 1)(k + 2) &= 0 \\ k_1 &= -1 \quad k_2 = -2 \end{aligned}$$

COMPLEMENTARY Function

$$y_c = Ae^{-x} + Be^{-2x}$$

### Step 3 : Finding a PARTICULAR solution of the complete equation (particular Integral) $y_p$

You need to consider  $f(x)$  and  $y_c$

$$\begin{aligned} y_c &= Ae^{-x} + Be^{-2x} \\ f(x) &= 2e^{-2x} \end{aligned} \quad \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 2y = 2e^{-2x}$$

|                                                              |                                                                       |                                                                                                              |
|--------------------------------------------------------------|-----------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------|
| $f(x) = ce^{\lambda x}$                                      | $y_c$ - contains $ae^{\lambda x}$ but <b>not</b> $axe^{\lambda x}$    | $y = axe^{\lambda x}$                                                                                        |
|                                                              | $y_c$ - contains $axe^{\lambda x}$                                    | $y = ax^2e^{\lambda x}$                                                                                      |
|                                                              | $y_c$ - does not contain $ae^{\lambda x}$ or $axe^{\lambda x}$        | $y = ae^{\lambda x}$                                                                                         |
| or<br>$f(x) = c \cos \lambda x$<br>$f(x) = c \sin \lambda x$ | $y_c - A \cos \lambda x + B \sin \lambda x$                           | $y = ax \sin \lambda x$ if $f(x) = c \cos \lambda x$<br>$y = ax \cos \lambda x$ if $f(x) = c \sin \lambda x$ |
|                                                              | $y_c$ - does <b>not</b> contain $A \cos \lambda x + B \sin \lambda x$ | $y = a \cos \lambda x + b \sin \lambda x$                                                                    |
| $f(x)$ polynomial f<br>degree n                              |                                                                       | $y = ax^n + bx^{n-1} + \dots$                                                                                |

Particular solution / integral

$$y_p = axe^{-2x}$$

### Step 4 : Finding a PARTICULAR solution of the complete equation (particular Integral) $y_p$

Differentiate  $y_p$  to find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  and substitute these into the original equation

$$y_p = axe^{-2x} \quad \frac{dy}{dx} = ae^{-2x} - 2axe^{-2x} \quad \frac{d^2y}{dx^2} = -2ae^{-2x} - 2(ae^{-2x} - 2axe^{-2x}) \\ = -4ae^{-2x} + 4axe^{-2x}$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}$$

$$\begin{aligned} -4ae^{-2x} + 4axe^{-2x} + 3(ae^{-2x} - 2axe^{-2x}) + 2axe^{-2x} &= 2e^{-2x} \\ -ae^{-2x} &= 2e^{-2x} \\ a &= -2 \\ y_p &= -2xe^{-2x} \end{aligned}$$

**Step 5 : General solution  $y = y_c + y_p$**

$$y = Ae^{-x} + Be^{-2x} - 2xe^{-2x}$$

**Step 6 : Use boundary values and General solution  $y = y_c + y_p$  to find the value of A and B**

## Harmonic Motion

- Motion in which acceleration is always towards a fixed point.
- Acceleration is proportional to the displacement from the centre of motion

**Notation :** Displacement  $x$      $v = \frac{dx}{dt} = \dot{x}$      $a = \frac{dx}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$

**Simple Harmonic Motion** often written as  $\ddot{x} = -\omega^2 x$  (a second order differential equation)

$$\ddot{x} = -\omega^2 x \quad \ddot{x} + \omega^2 x = 0$$

$$\text{Auxiliary equation } \lambda + \omega^2 = 0 \quad \lambda = \pm i\omega$$

$$\text{General solution } x = A\cos\omega t + B\sin\omega t \text{ can be written in the form } x = R\cos(\omega t - \varphi)$$

**EXAMPLE :** A particle of mass 2 kg is attached to spring and lies on a horizontal surface with negligible friction. The spring is stretched pulling the particle 0.4 m away from its starting position A. The spring constant is 8 N/m. The particle moves with an initial speed of  $0.3 \text{ ms}^{-1}$  towards A.

Find an equation for the displacement  $x$ , of the particle at time  $t$

$$F = ma \quad -8x = 2a$$

$$a = -4x$$

$$\ddot{x} = -4x \quad \ddot{x} + 4x = 0 \quad k = \pm 2i \quad \text{Auxiliary Equation}$$

$$\text{General equation } x = R\cos(2t - \varphi)$$

$$\text{When } t = 0 \quad x = 0.4 \text{ m} \quad \dot{x} = -0.3 \text{ ms}^{-1}$$

$$0.4 = R\cos(-\varphi) \quad R\cos(\varphi) = 0.4$$

$$-0.3 = -2R\sin(-\varphi) \quad 2R\sin(\varphi) = -0.3 \quad R\sin(\varphi) = -0.15$$

$$R^2 = (0.4)^2 + (-0.15)^2 \quad R = 0.427 \quad \tan(\varphi) = \frac{-0.15}{0.4} \quad \varphi = -0.359 \text{ radians (3 sf)}$$

$$x = 0.427\cos(2t + 0.359)$$

**Simple Harmonic Motion – damped motion**

- Factors in resistance that stops the oscillations
- Resistance is modelled as being proportionate to the velocity  $-r \frac{dx}{dt}$  ( $r$  is a non negative constant)
- acting in the direction opposite to motion

Resulting in the second order differential equation

$$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + kx = 0 \quad \text{leading to the auxiliary equation } m\lambda^2 + r\lambda + k = 0$$

**Use the discriminant of  $m\lambda^2 + r\lambda + k = 0$  to identify the type of damping**

$r^2 - 4mk > 0$  the system is **Heavily Damped** – comes to rest without oscillating

$x = Ae^{\alpha t} + Be^{\beta t}$  where  $\alpha$  and  $\beta$  are roots of the auxiliary equation  
A and B are unknown constants

$r^2 - 4mk = 0$  the system is **Critically Damped** – comes to rest in the quickest possible time

$x = (A + Bt)e^{-\omega t}$  where  $\omega = \sqrt{\frac{k}{m}}$  A and B are unknown constants

$r^2 - 4mk < 0$  the system is **Lightly Damped** - oscillations gradually get smaller

$x = Re^{pt}\cos(ut - \phi)$  where  $u = \sqrt{\frac{4mk - r^2}{2m}}$  and  $p = -\frac{r}{2m}$

### Simple Harmonic Motion Force Vibrations

$$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + kx = F(t)$$

## COUPLED FIRST ORDER DIFFERENTIAL EQUATIONS

$$\frac{dy}{dt} = mx + ny + f(t) \quad \text{and} \quad \frac{dx}{dt} = px + qy + g(t)$$

$$\text{Rearrange } \frac{dy}{dt} = mx + ny + f(t) \quad \text{to get } x = \frac{1}{m} \left[ \frac{dy}{dt} - ny - f(t) \right]$$

$$\text{Differentiate } \frac{dx}{dt} = \frac{1}{m} \left[ \frac{d^2y}{dt^2} - n \frac{dy}{dt} - \frac{d}{dt} f(t) \right]$$

$$\text{Substitute } \frac{1}{m} \left[ \frac{d^2y}{dt^2} - n \frac{dy}{dt} - \frac{d}{dt} f(t) \right] = p \left[ \frac{1}{m} \left[ \frac{dy}{dt} - ny - f(t) \right] \right] + qy + g(t)$$

$$\text{Rearrange } \frac{d^2y}{dt^2} - n \frac{dy}{dt} - \frac{d}{dt} f(t) = p \frac{dy}{dt} - pny - pf(t) + mgy + npg(t)$$

$$\frac{d^2y}{dt^2} - (p + n) \frac{dy}{dt} + (pn - mq)y = \frac{d}{dt} f(t) + pf(t) - npg(t)$$

Solve using the left-hand side

### EXAMPLE

$$\text{Solve } \frac{dy}{dt} = 2y - x \quad \text{and} \quad \frac{dx}{dt} = 5y + 4x \quad y(0) = -1 \quad x(0) = 1$$

$$\frac{dy}{dt} = 2y - x \quad x = 2y - \frac{dy}{dt} \quad \frac{dx}{dt} = 2 \frac{dy}{dt} - \frac{d^2y}{dt^2}$$

$$2 \frac{dy}{dt} - \frac{d^2y}{dt^2} = 5y + 4 \left( 2y - \frac{dy}{dt} \right)$$

$$\frac{d^2y}{dt^2} - 2 \frac{dy}{dt} - 4 \frac{dy}{dt} + 5y + 8y = 0$$

$$\frac{d^2y}{dt^2} - 6 \frac{dy}{dt} + 13y = 0$$

$$\text{Auxiliary equation } k^2 - 6k + 13 = 0$$

$$x = 3 \pm 2i$$

$$y = e^{3t}(A\cos 2t + B\sin 2t)$$

Using  $y(0) = -1 \quad -1 = A$

$$\frac{dy}{dt} = 3e^{3t}(A\cos 2t + B\sin 2t) + e^{3t}(-2A\sin 2t + 2B\cos 2t) \quad \frac{dy}{dt} = -2 - 1 = -3$$

$$-3 + 2B = -3$$

$$B = 0$$

$$y = -e^{3t} \cos 2t \quad \frac{dy}{dt} = -3e^{3t} \cos 2t + 2e^{3t} \sin 2t$$

$$x = 2y - \frac{dy}{dt} \quad x = 2(-e^{3t} \cos 2t) + 3e^{3t} \cos 2t - 2e^{3t} \sin 2t$$

$$x = e^{3t} \cos 2t - 2e^{3t} \sin 2t$$

## 12. Numerical Methods

**The mid-ordinate Rule (Numerical Integration)** Gives an approximation of the area under a graph.

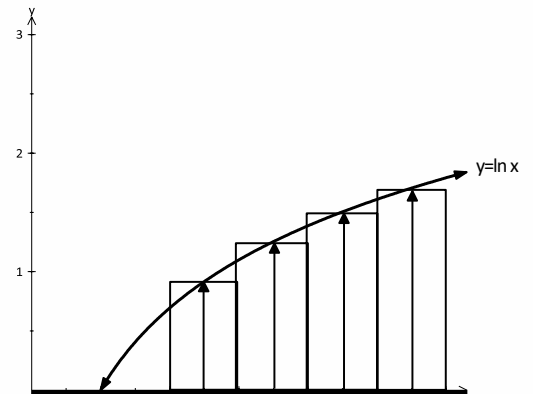
The area is divided into strips of equal width.

The value of the function halfway across each strip (the mid-ordinate) is calculated

Total area = width of strip x sum of mid-ordinates

| x   | Y      |
|-----|--------|
| 2.5 | ln 2.5 |
| 3.5 | ln 3.5 |
| 4.5 | ln 4.5 |
| 5.5 | ln 5.5 |

$$\begin{aligned} \text{Area} &= 1 \times (\ln 2.5 + \ln 3.5 + \ln 3.5 + \ln 5.5) \\ &= \ln (2.5 \times 3.5 \times 4.5 \times 5.5) \\ &= \ln 216.5625 \\ &= 5.38 \end{aligned}$$



### Simpson's Rule

Gives a more accurate approximation to the area under a graph.

An **even** number of strips of equal width are used.

The ordinates  $y_0, y_1, y_2, \dots$  are the values of the function on the vertical edges of the strips.

The area is given by

$$\frac{h}{3} (\text{sum of end ordinates} + 4(\text{sum of odd ordinates}) + 2(\text{sum of remaining even ordinates}))$$

### EULER'S FORMULA

**Numerical solution of first order differential equations**

- Problem given in the form  $\frac{dy}{dx} = f(x, y)$
  - Solution  $y(x)$
  - Boundary conditions  $y(x_0) = y_0$
  - Step length =  $h$  and  $x_{n+1} = x_n + h$
- $$\frac{dy}{dx} = f(x, y) \quad y_{r+1} = y_r + hf(x_r, y_r)$$

#### **EXAMPLE**

$$\frac{dy}{dx} = f(x, y) \quad \text{where } f(x, y) = x + 3 + \sin y \quad \text{and} \quad y(1) = 1$$

Use the Euler formula with  $h = 0.1$  to obtain an approximation to  $y(1.1)$  correct to 4 decimal places

$$y(x_0) = y_0 \quad x_0 = 1 \quad y_0 = 1$$

$$\begin{aligned} y(1.1) &= y_1 = 1 + 0.1(1 + 3 + \sin 1) \\ y_1 &= 1.4841 \end{aligned}$$

One improved method, sometimes call the mid-point formulae gives a better approximation – using 2 known values x values

$$y_{r+1} \approx y_{r-1} + 2hf(x_k, y_k)$$

You may need to use Euler's formula to find a second point if you are only given  $(x_0, y_0)$

Another improved methods, given in the question is the use the average of the gradients of the initial and end points of the curve.

$$y_{n-1} = y_n + \frac{1}{2}(k_1 + k_2) \quad \text{where} \quad k_1 = hf(x_0, y_0) \quad \text{and} \quad k_2 = hf(x_0 + h, y_0 + k_1)$$

The exam question will state which method to use.