

1. Proof

Proof by Induction

- Step 1 : Show that the rule works for the first value e.g. $n = 1$
- Step 2 : Make a statement : *Assume that the result holds for $n = k$*
- Step 3 : Show that the results holds for $n = k + 1$

Summation of a series

Example Prove by induction that $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$

Step 1 : Check for $n=1$

$$\text{LHS : } 1^2 = 1 \quad \text{RHS : } \frac{1}{6} \times 2 \times 3 = 1$$

So the formula is true for $n=1$

Step 2 : Assume that the result holds for the k^{th} step, when $n=k$ then $\sum_{r=1}^k r^2 = \frac{k}{6}(k+1)(2k+1)$

Step 3 : Show that the results holds for $k + 1$

$$\begin{aligned} \sum_{r=1}^{k+1} r^2 &= \sum_{r=1}^k r^2 + (k+1)^2 \quad \text{so} \quad \sum_{r=1}^{k+1} r^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2 \\ &= (k+1) \left(\left(\frac{k}{6} \right) (2k+1) + (k+1) \right) \\ &= \frac{(k+1)}{6} (2k^2 + 7k + 6) \\ &= \frac{(k+1)}{6} ((k+1) + 1)(2(k+1) + 1) \end{aligned}$$

This matches the formula with n replaced by $k+1$.

Take out any common factors as soon as you can as you work – not a good idea to expand and simplify fully before looking for factors

If the statement is true for $n = k$ then it is also true for $n = k+1$.
As the statement is true for $n = 1$ it must be true for all $n \geq 1$

Multiples

Example Prove by induction that $9^n - 1$ is a multiple of 8

Step 1 : Check for $n=1$ $9^1 - 1 = 8$ which is a multiple of 8

Step 2 : Assume that the result holds for a value $n = k$

Then $9^k - 1 = 8m$ where m is an integer

Step 3 : Show that the results holds for $k + 1$

$$\begin{aligned} 9^{k+1} - 1 &= 9^k \times 9 - 1 \quad \text{but} \quad 9^k = 8m + 1 \\ \text{so } 9^{k+1} - 1 &= (8m + 1) \times 9 - 1 \\ &= 72m + 9 - 1 \\ &= 8(9m + 1) \quad \text{so } 9^{k+1} - 1 \text{ is also a multiple of 8} \end{aligned}$$

If the statement is true for $n = k$ then it is also true for $n = k+1$.
As the statement is true for $n = 1$ it must be true for all $n \geq 1$

2. Complex Numbers

$$z = a + ib$$

real imaginary

$i = \sqrt{-1} \qquad i^2 = -1$

- Addition and Subtraction**

$$(2 + 3i) + (5 - 2i) = 7 + i \quad (\text{add/subtract real part then imaginary part})$$

- Multiplication** - multiply out the same way you would $(x - 2)(x + 4)$

EXAMPLE $(2 - 3i)(6 + 2i) = 12 + 4i - 18i - 6i^2$

$$= 12 - 14i + 6$$

$$= 18 - 14i$$

- Complex Conjugate z^***

If $z = a + ib$ then its complex conjugate is $z^* = a - ib$

always collect the 'real' and 'imaginary' parts before looking for the conjugate

- Solving Equations** - if two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

EXAMPLE Find z when $5z - 2z^* = 3 - 14i$

Let $z = x + iy$ and so $z^* = x - iy$

$$5(x + iy) - 2(x - iy) = 3 - 14i$$

$$3x + 7iy = 3 - 14i$$

Equating real : $3x = 3$ so $x = 1$

Equating imaginary : $7y = -14$ so $y = -2$ $z = 1 - 2i$

- Quadratic equations – $ax^2 + bx + c = 0$** (where a, b and c are real)

If $b^2 - 4ac < 0$ then the equation has 2 complex roots which will be **complex conjugates**

EXAMPLE Solve $4x^2 + 2x + 1 = 0$ $a = 4$ $b = 2$ $c = 1$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 4 \times 1}}{2 \times 4}$$

$$x = \frac{-2 \pm \sqrt{-12}}{8} \qquad x = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$$

- Cubic equations – $ax^3 + bx^2 + cx + d = 0$**

3 real roots

1 real root and 1 pair of complex conjugate roots

EXAMPLE

The cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots $x = 2$ and $x = 1 - 2i$ and a third root. Find the value of the constants a, b, c and d

Third root is $1 + 2i$ (complex conjugate)

$$(x - 2)(x - (1 - 2i))(x - (1 + 2i)) = ax^3 + bx^2 + cx + d$$

$$(x - 2)(x^2 - 2x + 5)$$

$$= x^3 - 2x^2 + 5x - 2x^2 + 4x - 10$$

$$= x^3 - 4x^2 + 9x + 10$$

$$a = 1, b = -4, c = 9, d = 10$$

$$(x - 1 + 2i)(x - 1 - 2i)$$

$$= x^2 - x - 2xi - x + 1 + 2i + 2xi - 2i + 4$$

$$= x^2 - 2x + 5$$

- Quartic equations – $ax^4 + bx^3 + cx^2 + dx + e = 0$**

4 real roots

2 real roots and 1 pair of complex conjugate roots

2 pair of complex conjugate roots

- **ARGAND DIAGRAMS**

$$z = x + iy$$

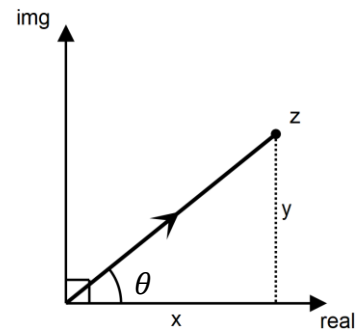
$$|z| = \sqrt{x^2 + y^2}$$

$$\arg(z) = \theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$\arg$ usually in radians where $-\pi < \theta < \pi$

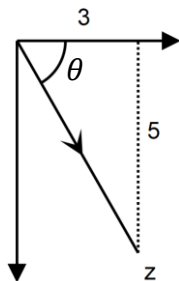
anti-clockwise from x axis gives a positive argument



- **Modulus-Argument form** $z = r(\cos \theta + i \sin \theta)$ where $r = |z|$ and $\theta = \arg z$

EXAMPLE

Write the complex number $z = 3 - 5i$ in modulus-argument form (radians to 2 d.p)



$$r = \sqrt{3^2 + 5^2}$$

$$= \sqrt{34}$$

$$\theta = \tan^{-1}\left(\frac{5}{3}\right)$$

$$\theta = -1.03$$

$$z = \sqrt{34}(\cos(-1.03) + i \sin(-1.03))$$

- **Multiplying and dividing complex numbers**

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

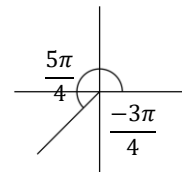
Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

You may need to adjust the arg to ensure that it lies within the range $-\pi < \theta < \pi$

$$\text{If } \theta_1 + \theta_2 = \frac{5\pi}{4}$$

$$= \frac{-3\pi}{4}$$



- **LOC**

Circles

$|z| = a$ circle radius a centre at the origin

$|z - z_1| = a$ circle radius a centre at z_1

$$|z - 5 - 3i| = 4$$

$$|z - (5 + 3i)| = 4 \quad \text{Centre } 5 + 3i \quad \text{radius } 4$$

Half Lines - straight lines which start at a specific point and only have one direction.

$\arg z = \theta$ represents the **half** line starting at 0 inclined at an angle θ to the positive x direction

$\arg(z - z_1) = \theta$ represents the **half** line starting at z_1 inclined at an angle θ to the positive x direction

EXAMPLE

For $\arg(z - 3 - 2i) = \frac{3\pi}{4}$ construct the locus of z

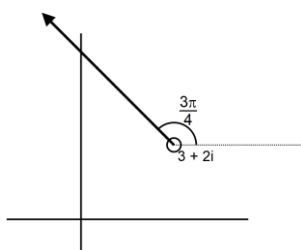
$$z = x + iy \quad \arg(z - (3 + 2i)) = \frac{3\pi}{4} \quad \arg(x + iy - (3 + 2i)) = \frac{3\pi}{4}$$

$$\tan^{-1}\left(\frac{y-2}{x-3}\right) = \frac{3\pi}{4}$$

$$\frac{y-2}{x-3} = -1$$

$$x + y = 5$$

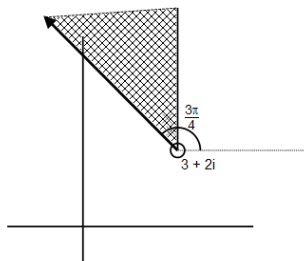
Restricted to $x < 3$ and $y > 2$



Regions

If the complex locus is an inequality then it will describe a region

$$\frac{\pi}{2} \leq \arg(z - 3 - 2i) \leq \frac{3\pi}{4}$$



Exponential form of complex numbers

Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$
 $-\pi < \theta < \pi$

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$-\pi < \theta < \pi$$

EXAMPLE: Express $\sqrt{3}(\cos(\frac{\pi}{3}) - i\sin(\frac{\pi}{3}))$ in the form $re^{i\theta}$

must be in the form $r(\cos\theta + i\sin\theta)$

Using $\cos(-\theta) = \cos\theta$
 $\sin(-\theta) = -\sin\theta$

$$\sqrt{3}(\cos(\frac{\pi}{3}) - i\sin(\frac{\pi}{3})) = \sqrt{3}(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})) = \sqrt{3}e^{-i\frac{\pi}{3}}$$

Multiplication and division where $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

For all integer values of n
 (can easily be proved by induction)

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Take care with negative values to show method

$$\begin{aligned} (\cos\theta - i\sin\theta)^n &= (\cos(-\theta) + i\sin(-\theta))^n \\ &= \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta - i\sin n\theta \end{aligned}$$

Make sure you show all of this

DO NOT write

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$$

Can be used to simplify complex calculations

EXAMPLE

Calculate $(1 + \sqrt{3})^6$ Let $z = 1 + \sqrt{3}$ $|z| = \sqrt{3}$

$$\arg z = \frac{\pi}{3}$$

$$z = \sqrt{3}(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}))$$

$$\begin{aligned} z^6 &= 27 \left(\cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right) \right) \\ &= 27 \end{aligned}$$

Multiple-angle trig functions to powers of trig functions

EXAMPLE

Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \quad (\text{real parts})$$

$$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$

$$= 4\cos^3\theta - 3\cos\theta$$

Powers of trig functions to multiple-angle trig functions

$$z^n = \cos n\theta + i \sin n\theta \quad z^{-n} = \cos(-n\theta) + i \sin(-n\theta) \\ = \cos n\theta - i \sin n\theta$$

$$\text{Leads to } \frac{z^n + z^{-n}}{2} = \cos n\theta \quad \frac{z^n - z^{-n}}{2i} = \sin n\theta$$

EXAMPLE

$$\text{Show that } \sin^4 \theta = \frac{\cos 4\theta - \cos 2\theta + 6}{8}$$

$$\text{Let } z = \cos \theta + i \sin \theta \quad 2i \sin \theta = z - z^{-1}$$

$$(2i \sin \theta)^4 = (2i)^4 \sin^4 \theta \\ = (z - z^{-1})^4 = z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4} \\ = ((z^4 + z^{-4}) - 4(z^2 + z^{-2})) + 6 \\ (2i)^4 \sin^4 \theta = 2 \cos 4\theta - 8 \cos 2\theta + 6$$

$$\sin^4 \theta = \frac{\cos 4\theta - \cos 2\theta + 6}{8}$$

Roots of Unity $z^n = 1$

The equation $z^n = 1$ has roots $\omega = e^{\frac{2k\pi i}{n}}$ $k = 0, 1, 2, \dots, (n-1)$

When plotted the roots are on a unit circle at intervals of $\frac{2\pi}{n}$ radians

EXAMPLE

Find the cube roots of unity

$$\omega = e^{\frac{2k\pi i}{n}} \quad k=0 \quad \omega = e^0 = 1 \\ k=1 \quad \omega = e^{\frac{2\pi i}{3}} \quad \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ k=2 \quad \omega = e^{\frac{4\pi i}{3}} \quad \left(\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

- For the n th roots of unit ($n > 1$) the sum of the roots is zero
- When plotted the roots are on a unit circle at intervals of $\frac{2\pi}{n}$ radians

Roots $z^n = \alpha$ has n solutions

EXAMPLE

Find all the solutions to $z^3 = 1 - i$

$$\text{Let } z = r e^{i\theta} \quad \text{so } z^3 = r^3 e^{i3\theta} \quad 1 - i = \sqrt{2} e^{i\left(-\frac{\pi}{4} + 2\pi k\right)}$$

$$r^3 e^{i3\theta} = \sqrt{2} e^{i\left(-\frac{\pi}{4} + 2\pi k\right)} \quad r^3 = \sqrt{2} \quad r = 2^{\frac{1}{6}}$$

$$e^{i3\theta} = e^{i\left(-\frac{\pi}{4} + 2\pi k\right)} \quad 3\theta = -\frac{\pi}{4} + 2\pi k$$

$$\theta = \frac{8\pi k - 1}{12}$$

$$\text{Roots are } 2^{\frac{1}{6}} e^{i\left(\frac{8\pi k - 1}{12}\right)} \quad k = 0, 1 \text{ and } 2$$

3. Matrices

- **Order of a matrix** 'm x n' where m is the number of rows and n the number of columns
(If order is correct for a particular operation then matrices are described as **conformable**)

Zero Matrix – all elements are 0

Identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad AI = IA = I$

Transpose of a matrix A^T write the rows as columns and the columns as rows $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Special result $(AB)^T = B^T A^T$

Determinant – every square matrix has a determinant
- if determinant is zero the matrix is **singular**

2 x 2 $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det M = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \det M \text{ is the area scale factor}$

3 x 3 $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \det M = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
 $\det M$ is the volume scale factor

Adding a scalar multiple of a row/column to another row/column does not change the determinant of the matrix

Factorising the determinant

Works in same way as factor theorem if $(x - a)$ is a factor of $|M|$ then $|M| = 0$ when $x = a$

EXAMPLE : Factorise the determinant of $\begin{bmatrix} a & b & c \\ bc & ca & ab \\ b+c & c+a & a+b \end{bmatrix}$

$$C1 = C1 - C2 \quad \begin{vmatrix} a-b & b & c \\ bc-ca & ca & ab \\ b-a & c+a & a+b \end{vmatrix} = (b-a) \begin{vmatrix} -1 & b & c \\ c & ca & ab \\ 1 & c+a & a+b \end{vmatrix}$$

$$C2 = C2 - C3 \quad (b-a) \begin{vmatrix} -1 & b-c & c \\ c & a(c-b) & ab \\ 1 & c-b & a+b \end{vmatrix} = (b-a)(c-b) \begin{vmatrix} -1 & -1 & c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix}$$

$$R1 = R1 + R3 \quad (b-a)(c-b) \begin{vmatrix} 0 & 0 & a+b+c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix} = (b-a)(c-b)(c-a)(a+b+c)$$

Inverses $A \times A^{-1} = I$ (If a matrix is singular $\det A = 0$ then it is not invertible (no inverse))

2 x 2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{the inverse } A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3 x 3 $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{the inverse } M^{-1} = \frac{1}{\det A} \begin{bmatrix} A & -D & G \\ -B & E & -H \\ C & -F & I \end{bmatrix}$

where $A = \begin{vmatrix} e & f \\ g & h \end{vmatrix} \quad B = \begin{vmatrix} d & f \\ g & i \end{vmatrix} \quad C = \begin{vmatrix} d & e \\ g & h \end{vmatrix} \quad D = \begin{vmatrix} b & c \\ g & i \end{vmatrix} \quad E = \begin{vmatrix} a & c \\ g & i \end{vmatrix} \dots$

Can be used to solve simultaneous equations

EXAMPLE Solve $x + 3y + 5z = -2$ $3x - 4y + 2z = 7$ $3x + 11y + 13z = 2$

$$\mathbf{AX} = \mathbf{B} \quad \begin{bmatrix} 1 & 3 & 5 \\ 3 & -4 & 2 \\ 3 & 11 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix} \quad \mathbf{A}^{-1} = \frac{1}{52} \begin{bmatrix} -74 & 16 & 26 \\ -33 & -2 & 13 \\ 45 & -2 & -13 \end{bmatrix}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{52} \begin{bmatrix} -74 & 16 & 26 \\ -33 & -2 & 13 \\ 45 & -2 & -13 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix} \quad x = 6 \quad y = 1.5 \quad z = -2.5 \quad \text{Unique solution}$$

Infinitely many solutions or no solution if $\det \mathbf{A} = 0$

- Addition and Subtraction** – must have the same order

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 4 & 7 \end{bmatrix}$$

- Multiplication** – for $\mathbf{AB}$ number of columns in $\mathbf{A}$ must equal number of rows in $\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 \times 2 + 2 \times 3 + 3 \times 4 & 1 \times -1 + 2 \times -2 + 3 \times 0 \\ 4 \times 2 + 5 \times 3 + 6 \times 4 & 4 \times -1 + 5 \times -2 + 6 \times 0 \end{bmatrix} \\ &= \begin{bmatrix} 20 & -5 \\ 47 & -14 \end{bmatrix} \end{aligned}$$

- Transformations– ORDER MATTERS**

Make sure you know the exact trig ratios

To calculate the coordinates of a point after a transformation

Multiply the Transformation Matrix by the coordinate

Find the position of point (2,1) after a stretch of Scale factor 5 parallel to the x-axis

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix} \quad (10,1)$$

To Identify a transformation from its matrix (consider the unit square)

Consider the points (1,0) and (0,1)

$$\mathbf{M} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{point (1,0)} \quad \mathbf{M} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \times 1 + 0 \times 0 \\ 0 \times 1 + 2 \times 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad (1,0) \rightarrow (4,0)$$

$$\text{point (0,1)} \quad \mathbf{M} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \times 0 + 0 \times 1 \\ 0 \times 0 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad (0,1) \rightarrow (0,2)$$

Stretch parallel to the x axis scale factor 4 and stretch parallel to the y axis scale factor 2

Standard Transformations

REFLECTIONS

$$\text{In the y axis} \quad \mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{In the x-axis} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{In the line } y = x \quad \mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{In the line } y = (\tan \theta)x \quad \mathbf{M} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

ENLARGEMENT

Scale factor k , centre $(0,0)$ $M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

STRETCH

Scale factor a parallel to the x -axis and Scale factor b parallel to the y -axis $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

ROTATION

Centre $(0,0)$ through θ **anti-clockwise** $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Centre $(0,0)$ through θ **clockwise** $M = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

ORDER MATTERS !!!! A figure is transformed by M_1 followed by M_2
Multiply $M_2 M_1$

WORKING IN 3D -same as in 2D – consider the points $(1,0,0)$ $(0,1,0)$ and $(0,0,1)$

Invariant Points - Points which are unaffected by a transformation and remain fixed

EXAMPLE Find the invariant points of the transformation with matrix $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad x + 2y = x \quad -y = y \quad (y = 0)$$

$y = 0$ the $x + 0 = x$ so all points $(x, 0)$ are invariant

Equation of the line of invariant points

The transformation matrix A is given by $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ find the equation of the line of invariant points passing through the origin $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad x' = 2x + 3y \quad y' = x + 4y \quad y' = mx' + c$

$$x + 4y = m(2x + 3y) + c$$

$$x + 4(mx + c) = m(2x + 3(mx + c)) + c$$

$$x(3m^2 - 2m - 1) + c(3m - 3) = 0$$

$$x(m - 1)(3m + 1) + c(3m - 3) = 0$$

If $m = 1$ c can take any value If $m = -\frac{1}{3}$ $c = 0$ (passing through the origin) $y = -\frac{1}{3}x$

4. Vectors (2D & 3D) $xi + yj + zk$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Magnitude $|xi + yj + zk| = \sqrt{x^2 + y^2 + z^2}$

Straight lines Vector Form $r = a + \lambda b$ **a**:position vector **b**:direction vector (parallel to the line)

Cartesian form : $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ (x,y,z) is a general point on the line

$$r = a + \lambda b \quad a = a_1i + a_2j + a_3k$$

$$b = b_1i + b_2j + b_3k$$

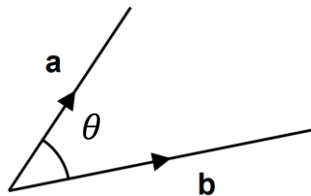
EXAMPLE

Find the vector equation of the line with cartesian equation $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-5}{4}$

$$r = (3i - j + 5k) + \lambda(2i + 3j + 4k)$$

Pairs of lines can - be parallel (look at direction vector)
 - intersect - a solution exists when solving simultaneously for all 3 variables
 - be skew (never meet) - no solution when solving simultaneously

Scalar product $a \cdot b = |a||b|\cos\theta$ where θ is the angle between vectors a and b when both directed away from the same point $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$



Perpendicular vectors have scalar product of zero

Finding the angle between 2 lines $\cos\theta = \frac{a \cdot b}{|a||b|}$

EXAMPLE: Find the acute angle between the lines $r = \begin{bmatrix} 0 \\ 7 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ and $r = \begin{bmatrix} -7 \\ 1 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$

Using the direction vectors

Let $a = 4i - 3j + k$ and $b = -3i + 2k$

$$a \cdot b = 4 \times -3 - 3 \times 0 + 1 \times 2 = -10 \quad |a| = \sqrt{26} \quad |b| = \sqrt{13}$$

$$\theta = \cos^{-1}\left(\frac{-10}{\sqrt{26}\sqrt{13}}\right) = 123^\circ \quad \text{Acute angle} = 57^\circ$$

Perpendicular distance from a point to a line - shortest path

EXAMPLE

Relative to a fixed origin O, the point A has position vector $(8i + 13j - 2k)$, the point B has position vector $(10i + 14j - 4k)$, and the point C has position vector $(9i + 9j + 6k)$.

Find the shortest distance from the point C to the line passing through A and B

$$\text{General point on line AB (point D)} = (8i + 13j - 2k) + \lambda(2i + j - 2k)$$

$$= (8 + 2\lambda)i + (13 + \lambda)j + (-2 - 2\lambda)k$$

$$CD = (9 - (8 + 2\lambda))i + (9 - (13 + \lambda))j + (6 - (-2 - 2\lambda))k$$

$$= (1 - 2\lambda)i + (-4 - \lambda)j + (8 + 2\lambda)k$$

Perpendicular if $(2i + j - 2k) \cdot (1 - 2\lambda)i + (-4 - \lambda)j + (8 + 2\lambda)k = 0$

$$2(1 - 2\lambda) + 1(-4 - \lambda) - 2(8 + 2\lambda) = 0$$

$$2 - 4\lambda - 4 - \lambda - 16 - 4\lambda = 0$$

$$-18 - 9\lambda = 0 \quad \lambda = -2$$

When $\lambda = -2$ $D = 4i + 11j + 2k$ $C = 9i + 9j + 6k$

$$\text{Distance between C and D} = \sqrt{(9 - 4)^2 + (9 - 11)^2 + (6 - 2)^2} \\ = 3\sqrt{5}$$

Perpendicular distance between parallel lines - pick a fixed point on one of the lines and complete as in the above example

Shortest distance between 2 skew lines

$$r_1 = a + \lambda b \quad \text{and} \quad r_2 = c + \mu d \quad \text{Shortest distance} = \left| \frac{(a-c) \cdot (b \times d)}{|b \times d|} \right|$$

Planes

Vector Form : $r = a + \lambda b + \mu c$ a is a fixed position vector

b and c are non-parallel vectors in the plane λ and μ scalars

EXAMPLE

Find the vector equation of the plane that passes through points A(1,1,2) B(2, 0,3) and C(0,-2,1)

$$a = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$$

$$r = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$$

Scalar Form $r \cdot n = a \cdot n = p$

r position vector of a general point on the plane

a position vector of a fixed point on the plane

n normal to the plane

p a scalar constant

Cartesian form

$$r \cdot n = p \quad ax + by + cz + d = 0$$

$$p = -d$$

EXAMPLE

Find the equation of the plane with normal vector $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ that contains point (3, 0, -2)

$$r \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = 3 + 0 + 4 = 7 \quad x + 2y - 2z = 7$$

Angle between a line and a plane

$$\text{Plane } r \cdot n = p \quad \text{and line : } r = a + \lambda b \quad \sin \phi = \left| \frac{n \cdot b}{|n||b|} \right|$$

Angle between two planes

n_1 is the normal to plane A n_2 is the normal to plane B

The acute angle between the normals = the acute angle between the planes

$$\text{Angle between the two planes is } 180 - \theta \quad \text{where } \cos \theta = \left| \frac{n_1 \cdot n_2}{|n_1||n_2|} \right|$$

EXAMPLE

Calculate the obtuse angle between the planes

$$r \cdot (13i - 9j + 5k) = 8$$

$$r \cdot (2j + 3k) = 4$$

$$n_1 = 13i - 9j + 5k \quad n_2 = 2j + 3k$$

$$\cos\theta = \left| \frac{(13i-9j+5k)(2j+3k)}{\sqrt{13^2+(-9)^2+5^2} \times \sqrt{2^2+3^2}} \right| = \left| \frac{-3}{\sqrt{275}\sqrt{13}} \right|$$

$$\theta = 87.1^\circ \text{ so the obtuse angle is } 92.9^\circ$$

Point of intersection of a line and a plane

EXAMPLE

Find the point of intersection of the line $r = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$ and the plane $r \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 13$

$$\text{Line } r = \begin{bmatrix} 1+4\lambda \\ 2+\lambda \\ 1-3\lambda \end{bmatrix} \quad \begin{bmatrix} 1+4\lambda \\ 2+\lambda \\ 1-3\lambda \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 13$$

$$3 + 12\lambda - 2 - \lambda + 2 - 6\lambda = 13$$

$$3 + 5\lambda = 13$$

$$\lambda = 2$$

$$\text{Point of intersection} = (9, 4, -5)$$

Intersecting Planes

Any two (non parallel planes) meet along a line – **line of intersection**

EXAMPLE

Find a cartesian equation of the line of intersection of the two planes $r \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 11$ and $r \cdot \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = -7$

$$x - 2y + 3z = 11 \quad 2x + 2y - 2z = -7$$

$$(1) + (2) \quad 3x + z = 4 \quad z = 4 - 3x \quad x = \frac{4-z}{3}$$

$$\text{Subs into (2)} \quad 2x + 2y - 2(4 - 3x) = -7$$

$$8x + 2y = 1 \quad x = \frac{1-2y}{8}$$

$$x = \frac{1-2y}{8} = \frac{4-z}{3}$$

3 non-parallel planes can

- intersect at a single point
- all meet at the same line
- meet along different lines

Perpendicular distance between a point and a plane

$$\text{Point } a = a_1i + a_2j + a_3k \text{ and plane } r \cdot n = p \quad \text{Perpendicular distance} = \frac{|a \cdot n - p|}{|n|}$$

Vector Product

$$a \times b = |a||b|\sin\theta \hat{n} \quad \theta \text{ is the angle between the 2 vectors when they are directed away from the same point}$$

$$\hat{n} \text{ is a unit vector perpendicular to } a \text{ and } b \quad (\text{in a right-handed sense})$$

If 2 vectors are parallel then $a \times b = 0$

The vector product is the determinant of a 3 x 3 matrix

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Vector product to find the area of a triangle (based on $\frac{1}{2}ab\sin C$)

When θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ pointing away from a point the area of the triangle formed is

$$\text{Area} = \frac{1}{2} |\mathbf{A} \times \mathbf{B}|$$

Example

Find the area of the triangle formed by $\mathbf{a} = 2\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 2 & -6 & 4 \end{vmatrix} \quad |(0-6)\mathbf{i} - (8-2)\mathbf{j} + (-12)\mathbf{k}| = \sqrt{(-6)^2 + (6)^2 + (-12)^2}$$

$$= \sqrt{216}$$

$$\text{Area} = \frac{\sqrt{216}}{2} = 3\sqrt{6}$$

5. Polynomials

Quadratics

For a quadratic equation $\mathbf{ax^2 + bx + c = 0}$ with roots α and β

$$\text{Sum of the roots } \alpha + \beta = -\frac{b}{a} \quad \text{Product of roots } \alpha\beta = \frac{c}{a}$$

Once the value $\alpha + \beta$ and $\alpha\beta$ have been found, new quadratic equations can be formed with roots :

Roots	Sum of roots	Product of roots
α^2 and β^2	$(\alpha + \beta)^2 - 2\alpha\beta$	$(\alpha\beta)^2$
α^3 and β^3	$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$	$(\alpha\beta)^3$
$\frac{1}{\alpha}$ and $\frac{1}{\beta}$	$\frac{\alpha + \beta}{\alpha\beta}$	$\frac{1}{\alpha\beta}$

The new equation becomes $\mathbf{x^2 - (sum of new roots)x + (product of new roots) = 0}$

EXAMPLE

The roots of the quadratic equation $3x^2 + 4x - 1 = 0$ are α and β

Determine a quadratic equation with integer coefficients which has roots $\alpha^3\beta$ and $\alpha\beta^3$

$$\text{Step 1 : } \alpha + \beta = -\frac{4}{3} \quad \text{and } \alpha\beta = -\frac{1}{3}$$

$$\alpha + \beta = -\frac{4}{3} \quad \alpha\beta = -\frac{1}{3}$$

$$\begin{aligned} \text{Step 2 : Sum of new roots } \alpha^3\beta + \alpha\beta^3 &= \alpha\beta(\alpha^2 + \beta^2) \\ &= -\frac{1}{3}((\alpha + \beta)^2 - 2\alpha\beta) \\ &= -\frac{1}{3}\left(\left(-\frac{4}{3}\right)^2 + \frac{2}{3}\right) \\ &= -\frac{22}{27} \end{aligned}$$

$$\begin{aligned} \text{Step 3 : Product of new roots } \alpha^3\beta\alpha\beta^3 &= (\alpha\beta)^4 \\ &= \frac{1}{81} \end{aligned}$$

$$\text{Step 4 : Form the new equation } x^2 + \frac{22}{27}x + \frac{1}{81} = 0 \quad 81x^2 + 66x + 1 = 0$$

CUBIC EQUATIONS $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ

$$\alpha + \beta + \gamma = -\frac{b}{a} \quad \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} \quad \alpha\beta\gamma = -\frac{d}{a}$$

QUARTIC EQUATIONS (and beyond....) $ax^4 + bx^3 + cx^2 + dx + e = 0$

$$\sum \alpha = -\frac{b}{a} \quad \sum \alpha\beta = \frac{c}{a} \quad \sum \alpha\beta\gamma = -\frac{d}{a} \quad \sum \alpha\beta\gamma\delta = \frac{e}{a}$$

EXAMPLE

The quartic equation $x^4 + 2x^3 + 14x + 15 = 0$ has one root equal to $1 + 2i$. Find the other three roots.

As the coefficients are real, it follows that $1 - 2i$ is also a root (complex conjugate)

Hence $(x - (1 - 2i))(x - (1 + 2i))$ is a quadratic factor of the quartic

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + ax + b)$$

Equating coefficients: $2 = a - 2$

$$a = 4 \quad 15 = 5b$$

$$b = 3$$

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + 4x + 3)$$

the four roots are $1 + 2i$, $1 - 2i$, -3 , and -1 .

6. Series

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

Key Points

Always look carefully at the limits

$$\sum_{r=1}^n 1 = n$$

$$\sum_{r=n+1}^{2n} r^3 = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^n r^3$$

$$\sum 5r^2 = 5 \sum r^2$$

The method of differences

- Often seen as a 2 part question – i) show that ii) hence find
- ALWAYS show clearly the cancelling process

Example

Express $\frac{1}{r(r+2)}$ in partial fractions.

Hence show that $\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$

$$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$$

$$A(r+2) + Br = 1$$

$$A = \frac{1}{2} \quad B = -\frac{1}{2}$$

$$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

$$\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{1}{2(1)} - \frac{1}{2(3)} + \frac{1}{2(2)} - \frac{1}{2(4)} + \frac{1}{2(3)} - \frac{1}{2(5)} + \frac{1}{2(4)} - \frac{1}{2(6)} + \dots + \frac{1}{2(n-1)} - \frac{1}{2(n+1)} + \frac{1}{2(n)} - \frac{1}{2(n+2)}$$

$$\begin{aligned}
&= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \\
&= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)} \\
&= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)} \\
&= \frac{n(3n+5)}{4(n+1)(n+2)}
\end{aligned}$$

7. Hyperbolic Functions

You need to learn these as they are not given in the formula booklet

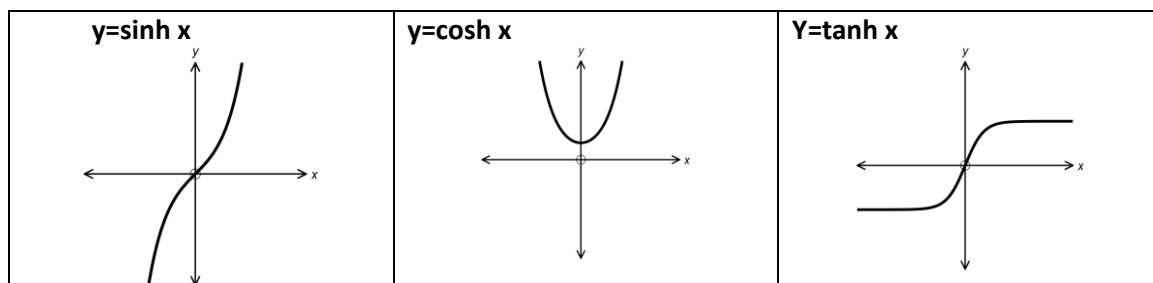
$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

EXAMPLE Find the exact value of $\cosh(\ln 4)$

$$\begin{aligned}
\cosh(\ln 4) &= \frac{1}{2}(e^{\ln 4} + e^{-(\ln 4)}) \\
&= \frac{1}{2}\left(4 + \frac{1}{4}\right) = \frac{17}{8}
\end{aligned}$$

Identity: $\cosh^2 x - \sinh^2 x = 1$

Graphs of hyperbolic functions



Inverse hyperbolic functions – given in the formula booklet

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \quad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad \tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

$x \geq 1$
 $|x| < 1$

EXAMPLE: Solve $\operatorname{artanh} x = \ln 4$

$$\begin{aligned}
\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) &= \ln 4 & \ln \left(\frac{1+x}{1-x} \right) &= \ln 16 \\
\frac{1+x}{1-x} &= 16 \\
1+x &= 16 - 16x \\
x &= \frac{15}{17}
\end{aligned}$$

Reciprocal hyperbolic functions Use exponential forms to solve equations

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}} \quad \operatorname{sech} x = \frac{2}{e^x + e^{-x}} \quad \operatorname{coth} x = \frac{e^{2x} + 1}{e^{2x} - 1}$$

Identities: $\operatorname{coth}^2 x - 1 = \operatorname{cosech}^2 x$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

EXAMPLE

$$\text{Solve } 2\coth^2 x - 5\operatorname{cosech} x = 5$$

(answers in exact form)

$$\text{Using } \coth^2 x = \operatorname{cosech} x + 1$$

$$2(\operatorname{cosech} x + 1) - 5\operatorname{cosech} x = 5$$

$$2\operatorname{cosech} x + 2 - 5\operatorname{cosech} x - 5 = 0$$

$$2\operatorname{cosech} x - 5\operatorname{cosech} x - 3 = 0$$

$$(2\operatorname{cosech} x + 1)(\operatorname{cosech} x - 3) = 0$$

$$\operatorname{cosech} x = -\frac{1}{2} \quad \operatorname{cosech} x = 3$$

$$\begin{aligned} \sinh x &= -2 & x &= \sinh^{-1}(-2) \\ & & &= \ln(-2 + \sqrt{(-2)^2 + 1}) \\ & & &= \ln(-2 + \sqrt{5}) \end{aligned}$$

$$\begin{aligned} \sinh x &= \frac{1}{3} & x &= \sinh^{-1}\left(\frac{1}{3}\right) \\ & & &= \ln\left(\frac{1}{3} + \sqrt{\left(\frac{1}{3}\right)^2 + 1}\right) \\ & & &= \ln\left(\frac{1}{3} + \sqrt{\frac{10}{9}}\right) \end{aligned}$$

Calculus with hyperbolic functions sometimes easier to integrate or differentiate in exponential form

You need to learn these $\frac{d}{dx} \sinh x = \cosh x$ $\frac{d}{dx} \cosh x = \sinh x$ (tan given in formula book)

$$\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \coth x \quad \frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x \quad \frac{d}{dx} \coth x = -\operatorname{cosec}^2 \theta$$

Derivatives of the inverse functions are given in the formula booklet

Look out for some of the standard integrals which can be used with a little substitution

EXAMPLE

$$\begin{aligned} \text{Find } \int \frac{1}{\sqrt{x^2 + 14x + 24}} dx \\ = \int \frac{1}{\sqrt{(x+7)^2 - 25}} dx \end{aligned}$$

$$\text{Let } x + 7 = 5\cosh u \quad u = \cosh^{-1}\left(\frac{x+7}{5}\right) \quad \frac{dx}{du} = 5\sinh u$$

$$\int \frac{1}{\sqrt{(5\cosh u)^2 - 25}} \times 5\sinh u \, du$$

$$= \int \frac{5\sinh u}{\sqrt{25\cosh^2 u - 25}} \, du$$

$$= \int \frac{5\sinh u}{5\sqrt{\cosh^2 u - 1}} \, du = \int \frac{5\sinh u}{5\sqrt{\sinh^2 u}} \, du = \int 1 \, du = u + c$$

$$\int \frac{1}{\sqrt{x^2 + 14x + 24}} \, dx = \cosh^{-1}\left(\frac{x+7}{5}\right) + c$$

8. Further Calculus

Maclaurin Series standard series given in the formula booklet

Example

Find the Maclaurin expansion of $f(x) = \ln(1 + 2x)$ up to the x^3 term.

State the range of values for which it is valid

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} & \ln(1+2x) &= (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} \\ & & &= 2x - 2x^2 + \frac{8x^3}{3} \end{aligned}$$

$$\ln(1+x) \text{ valid for } -1 < x < 1 \quad \ln(1+2x) \text{ valid for } -1 < 2x < 1 \quad \text{so } -\frac{1}{2} < x < \frac{1}{2}$$

Deriving a Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} \dots \dots \dots$$

Assumption

- $f(x)$ can be expressed as $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$
- The series can be differentiated term by term
- The function $f(x)$ and all of its derivatives exist at $x = 0$

If you are finding products of expansions check the range of values for which each is valid

Improper Integrals

An integral is improper if

- One or both of the limits is infinity
- the integrand is undefined at one of the limits or somewhere in between the limits

Replace the 'undefined' limit with n and allow n to approach the limit once the function has been integrated

EXAMPLE

$$\text{Determine } \int_1^\infty \frac{1}{x^2} dx \quad \int_1^n \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^n \\ = -\frac{1}{n} + 1$$

As $n \rightarrow \infty$ the value of the integral $\rightarrow 1$

If an undefined value between the limits

$$\text{split the integral e.g. } \int_{-2}^2 \frac{1}{x^2} dx \\ \int_{-2}^2 \frac{1}{x^2} dx = \int_{-2}^0 \frac{1}{x^2} dx + \int_0^2 \frac{1}{x^2} dx$$

Known Results – learn these

$$\lim_{x \rightarrow \infty} x^n e^{-x} = 0 \quad \lim_{x \rightarrow 0^+} x^n \ln x = 0$$

Volumes of revolution

Rotation about the x – axis $\int_{x=a}^{x=b} \pi y^2 dx$

Rotation about the y – axis $\int_{y=a}^{y=b} \pi x^2 dy$

Don't forget the π in your answer

Mean value of a Function

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$$

Integration with partial fractions - learn the different forms

$$\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$$\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-a)^2}$$

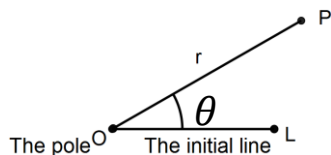
$$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

2 special forms to look for (both given in the formula booklet)

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \quad |x| < a$$

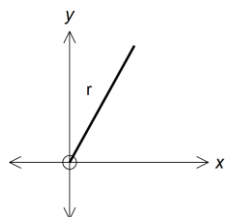
9. Polar Coordinates



P has polar coordinate (r, θ)



Polar – Cartesian

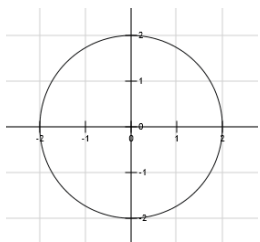
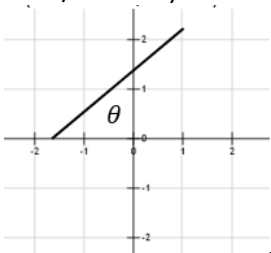
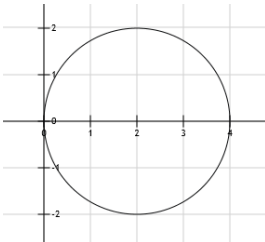
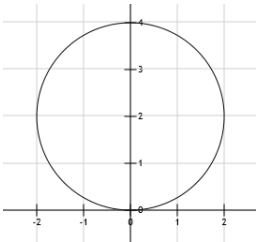


$$x = r \cos \theta \quad y = r \sin \theta$$

$$r^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}$$

Plotting Curves and Graphs

$r = a$ circle centre (0,0) radius a 	$\theta = \alpha$ Straight line (semi-infinite/half line) 	$r = 2a \cos \theta$ 	$r = 2a \sin \theta$ 
--	---	--	---

Hints for sketching curves

- if r is expressed as a function of $\cos$ only - **symmetrical about the initial line**
- if r is expressed as a function of $\sin$ only - **symmetrical about the line $\theta = \frac{\pi}{2}$**
- if $r \rightarrow 0$ as $\theta \rightarrow \infty$ then the line $\theta = \alpha$ is a tangent to the curve at the pole
- **NEGATIVE VALUES** of r are not allowed – if $r < 0$ in the interval $\alpha < \theta < \beta$ then there is no curve in this interval

Integrating Polar Curves

$$\int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

r must be non-negative and defined for $\alpha < \theta < \beta$

MAKE SURE YOU CAN INTEGRATE TRIG FUNCTIONS

You may need to use

$$\int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta$$

$$\int \sin^2 \theta = \int \frac{1}{2} (1 - \cos 2\theta) = \frac{1}{2} (\theta - \frac{1}{2} \sin 2\theta)$$

$$\int \sin^2 2\theta d\theta = \int \frac{1}{2} (1 - \cos 4\theta) d\theta = \frac{1}{2} (\theta - \frac{1}{4} \sin 4\theta)$$

$$\int \sin^n \theta \cos \theta d\theta = \frac{1}{n+1} \sin^{n+1} \theta$$

10. Differential Equations

First order differential equations

Linear 1st order differential equation $\frac{dy}{dx} - \frac{y}{x} = 2x^2$ (dependent variable 'y' is linear)

A **General solution** has unknown variables and represents a family of solutions e.g. $y = Ae^{x^2}$

To find a **Particular solution** boundary or initial conditions are used to evaluate unknown variables

e.g. when $y = 2$ $x = 0$ or $y(0) = 2$ both mean the same thing

METHOD 1 : Separating the

$$\frac{dy}{dx} = 3x^2y$$

$$\frac{1}{y} dy = 3x^2 dx$$

$$\int \frac{1}{y} dy = \int 3x^2 dx$$

$$\ln|y| = x^3 + c$$

$$y = e^{x^3+c}$$

$$y = Ae^{x^3}$$

METHOD 2 : Using an Integrating Factor

Step 1 : Check that the equation is in the form $\frac{dy}{dx} + P(x)y = Q(x)$

$$x \frac{dy}{dx} - 2y = x^3 \quad \frac{dy}{dx} - \frac{2y}{x} = x^2$$

$$P(x) = -\frac{2}{x} \quad Q(x) = x^2$$

Step 2 : Find the 'Integrating Factor' $I(x) = e^{\int P(x)dx}$

$$\int P(x)dx = \int \frac{-2}{x} dx = -2 \ln|x|$$

$$I(x) = e^{-2 \ln|x|} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

The left hand side is the result of differentiating $I(x)y$ using the product rule

Step 3 : Multiply all terms by the Integrating Factor

$$\frac{1}{x^2} \frac{dy}{dx} - \frac{2y}{x^3} = 1$$

$$\frac{d}{dx} \left(\frac{y}{x^2} \right) = 1$$

$$\frac{y}{x^2} = \int 1 dx$$

$$\frac{y}{x^2} = x + c$$

$$y = x^3 + cx^2$$

$$\frac{d}{dx} (I(x)y) = I(x)Q(x)$$

MAKE SURE YOU SHOW THIS

Don't forget to include the 'c' in any rearrangement to isolate y
e.g. cx^2

Step 4 : You have found the GENERAL solution – if necessary use values given to find a PARTICULAR solution

Second order differential equations

Solving equations of the form $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$

Step 1 : Solve the auxiliary equation $ak^2 + bk + c = 0$

Solve

Find the general solution to $2 \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} + 5y = 0$

$$2k^2 + 6k + 5 = 0 \quad (\text{Auxiliary equation})$$

$$k = \frac{-6 \pm \sqrt{6^2 - 4 \times 2 \times 5}}{2 \times 2}$$

$$k = -\frac{3}{2} \pm \frac{1}{2}i$$

3 possibilities for values

k_1 and k_2 real and unequal	$k_1 = k_2$ equal roots	$K_1 = p+qi$ $K_2 = p-qi$ (non real)
$y = Ae^{k_1x} + Be^{k_2x}$	$y = e^{k_1x}(A + Bx)$	$y = e^{px}(A\cos qx + B\sin qx)$

Exceptional case if solving $\frac{d^2y}{dx^2} + n^2y = 0$ $y = A\cos nx + B\sin nx$

Step 2 : Use the chosen form of y for the general solution

General solution

$$y = e^{-\frac{3}{2}x} \left(A\cos \frac{1}{2}x + B\sin \frac{1}{2}x \right)$$

Solving equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

Step 1 & 2 : Solve the auxiliary equation $ak^2 + bk + c = 0$ (as above)

This will find the COMPLEMENTARY Function , CF, y_c

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}$$

Auxiliary equation

$$\begin{aligned} k^2 + 3k + 2 &= 0 \\ (k+1)(k+2) &= 0 \\ K_1 &= -1 \quad K_2 = -2 \end{aligned}$$

COMPLEMENTARY Function

$$y_c = Ae^{-x} + Be^{-2x}$$

Step 3 : Finding a PARTICULAR solution of the complete equation (particular Integral) y_p

You need to consider $f(x)$ and y_c

$$\begin{aligned} y_c &= Ae^{-x} + Be^{-2x} \\ f(x) &= 2e^{-2x} \end{aligned} \quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}$$

$f(x) = ce^{\lambda x}$	y_c - contains $ae^{\lambda x}$ but not $axe^{\lambda x}$	$y = axe^{\lambda x}$
	y_c - contains $axe^{\lambda x}$	$y = ax^2e^{\lambda x}$
	y_c - does not contain $ae^{\lambda x}$ or $axe^{\lambda x}$	$y = ae^{\lambda x}$
or $f(x) = c\sin \lambda x$	$y_c - A\cos \lambda x + B\sin \lambda x$	$y = ax\sin \lambda x$ if $f(x) = c\cos \lambda x$ $y = ax\cos \lambda x$ if $f(x) = c\sin \lambda x$
	y_c - does not contain $A\cos \lambda x + B\sin \lambda x$	$y = a\cos \lambda x + b\sin \lambda x$
$f(x)$ polynomial f degree n		$y = ax^n + bx^{n-1} + \dots$

Particular solution / integral

$$y_p = axe^{-2x}$$

Step 4 : Finding a PARTICULAR solution of the complete equation (particular Integral) y_p

Differentiate y_p to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and substitute these into the original equation

$$y_p = axe^{-2x} \quad \frac{dy}{dx} = ae^{-2x} - 2axe^{-2x} \quad \frac{d^2y}{dx^2} = -2ae^{-2x} - 2(ae^{-2x} - 2axe^{-2x}) \\ = -4ae^{-2x} + 4axe^{-2x}$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{-2x}$$

$$-4ae^{-2x} + 4axe^{-2x} + 3(ae^{-2x} - 2axe^{-2x}) + 2axe^{-2x} = 2e^{-2x}$$

$$-ae^{-2x} = 2e^{-2x}$$

$$a = -2$$

$$y_p = -2xe^{-2x}$$

Step 5 : General solution $y = y_c + y_p$

$$y = Ae^{-x} + Be^{-2x} - 2xe^{-2x}$$

Step 6 : Use boundary values and General solution $y = y_c + y_p$ to find the value of A and B

Harmonic Motion

- Motion in which acceleration is always towards a fixed point.
- Acceleration is proportional to the displacement from the centre of motion

Notation : Displacement x $v = \frac{dx}{dt} = \dot{x}$ $a = \frac{dx}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$

Simple Harmonic Motion often written as $\ddot{x} = -\omega^2 x$ (a second order differential equation)

$$\ddot{x} = -\omega^2 x \quad \ddot{x} + \omega^2 x = 0$$

Auxiliary equation $\lambda + \omega^2 = 0$ $\lambda = \pm i\omega$

General solution $x = A\cos\omega t + B\sin\omega t$ can be written in the form $x = R\cos(\omega t - \phi)$

EXAMPLE : A particle of mass 2 kg is attached to spring and lies on a horizontal surface with negligible friction. The spring is stretched pulling the particle 0.4 m away from its starting position A. The spring constant is 8 N/m. The particle moves with an initial speed of 0.3 ms^{-1} towards A.

Find an equation for the displacement x , of the particle at time t

$$F = ma \quad -8x = 2a$$

$$a = -4x$$

$$\ddot{x} = -4x \quad \ddot{x} + 4x = 0 \quad k = \pm 2i \quad \text{Auxiliary Equation}$$

General equation $x = R\cos(2t - \phi)$

When $t = 0$ $x = 0.4 \text{ m}$ $\dot{x} = -0.3 \text{ ms}^{-1}$

$$0.4 = R\cos(-\phi) \quad R\cos(\phi) = 0.4$$

$$-0.3 = -2R\sin(-\phi) \quad 2R\sin(\phi) = -0.3 \quad R\sin(\phi) = -0.15$$

$$R^2 = (0.4)^2 + (-0.15)^2 \quad R = 0.427$$

$$\tan(\phi) = \frac{-0.15}{0.4} \quad \phi = -0.359 \text{ radians (3 sf)}$$

$$x = 0.427\cos(2t + 0.359)$$

Simple Harmonic Motion – damped motion

- Factors in resistance that stops the oscillations
- Resistance is modelled as being proportionate to the velocity $-r\frac{dx}{dt}$ (r is a non negative constant)

- acting in the direction opposite to motion

Resulting in the second order differential equation

$$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + kx = 0 \quad \text{leading to the auxiliary equation } m\lambda^2 + r\lambda + k = 0$$

Use the discriminant of $m\lambda^2 + r\lambda + k = 0$ to identify the type of damping

$r^2 - 4mk > 0$ the system is **Heavily Damped** – comes to rest without oscillating

$$x = Ae^{\alpha t} + Be^{\beta t} \quad \text{where } \alpha \text{ and } \beta \text{ are roots of the auxiliary equation}$$

A and B are unknown constants

$r^2 - 4mk = 0$ the system is **Critically Damped** – comes to rest in the quickest possible time

$$x = (A + Bt)e^{-\omega t} \quad \text{where } \omega = \sqrt{\frac{k}{m}} \quad \text{A and B are unknown constants}$$

$r^2 - 4mk < 0$ the system is **Lightly Damped** - oscillations gradually get smaller

$$x = Re^{pt} \cos(ut - \phi) \quad \text{where } u = \sqrt{\frac{4mk - r^2}{2m}} \text{ and } p = -\frac{r}{2m}$$

Simple Harmonic Motion Force Vibrations

$$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + kx = F(t)$$

COUPLED FIRST ORDER DIFFERENTIAL EQUATIONS

$$\frac{dy}{dt} = mx + ny + f(t) \quad \text{and} \quad \frac{dx}{dt} = px + qy + g(t)$$

$$\text{Rearrange } \frac{dy}{dt} = mx + ny + f(t) \quad \text{to get } x = \frac{1}{m} \left[\frac{dy}{dt} - ny - f(t) \right]$$

$$\text{Differentiate } \frac{dx}{dt} = \frac{1}{m} \left[\frac{d^2y}{dt^2} - n \frac{dy}{dt} - \frac{d}{dt} f(t) \right]$$

$$\text{Substitute } \frac{1}{m} \left[\frac{d^2y}{dt^2} - n \frac{dy}{dt} - \frac{d}{dt} f(t) \right] = p \left[\frac{1}{m} \left[\frac{dy}{dt} - ny - f(t) \right] \right] + qy + g(t)$$

$$\text{Rearrange } \frac{d^2y}{dt^2} - n \frac{dy}{dt} - \frac{d}{dt} f(t) = p \frac{dy}{dt} - pny - pf(t) + mqy + npg(t)$$

$$\frac{d^2y}{dt^2} - (p + n) \frac{dy}{dt} + (pn - mq)y = \frac{d}{dt} f(t) + pf(t) - npg(t)$$

Solve using the left hand side

EXAMPLE

Solve $\frac{dy}{dt} = 2y - x$ and $\frac{dx}{dt} = 5y + 4x$ $y(0) = -1$ $x(0) = 1$

$$\frac{dy}{dt} = 2y - x \quad x = 2y - \frac{dy}{dt} \quad \frac{dx}{dt} = 2\frac{dy}{dt} - \frac{d^2y}{dt^2}$$

$$2\frac{dy}{dt} - \frac{d^2y}{dt^2} = 5y + 4\left(2y - \frac{dy}{dt}\right)$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 4\frac{dy}{dt} + 5y + 8y = 0$$

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 13y = 0 \quad \text{Auxiliary equation } k^2 - 6k + 13 = 0$$

$$x = 3 \pm 2i$$

$$y = e^{3t}(A\cos 2t + B\sin 2t)$$

Using $y(0) = -1$ $-1 = A$

$$\frac{dy}{dt} = 3e^{3t}(A\cos 2t + B\sin 2t) + e^{3t}(-2A\sin 2t + 2B\cos 2t) \quad \frac{dy}{dt} = -2 - 1 = -3$$

$$-3 + 2B = -3$$

$$B = 0$$

$$y = -e^{3t} \cos 2t \quad \frac{dy}{dt} = -3e^{3t} \cos 2t + 2e^{3t} \sin 2t$$

$$x = 2y - \frac{dy}{dt} \quad x = 2(-e^{3t} \cos 2t) + 3e^{3t} \cos 2t - 2e^{3t} \sin 2t$$

$$x = e^{3t} \cos 2t - 2e^{3t} \sin 2t$$