

1. Proof

Proof by Induction

- Step 1 : Show that the rule works for the first value e.g. $n = 1$
- Step 2 : Make a statement : *Assume that the result holds for $n = k$*
- Step 3 : Show that the results holds for $n = k + 1$

Summation of a series

Example Prove by induction that $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$

Step 1 : Check for $n=1$

$$\text{LHS : } 1^2 = 1 \quad \text{RHS : } \frac{1}{6} \times 2 \times 3 = 1$$

So the formula is true for $n=1$

Step 2 : Assume that the result holds for the k^{th} step, when $n=k$ then $\sum_{r=1}^k r^2 = \frac{k}{6}(k+1)(2k+1)$

Step 3 : Show that the results holds for $k + 1$

$$\begin{aligned} \sum_{r=1}^{k+1} r^2 &= \sum_{r=1}^k r^2 + (k+1)^2 \quad \text{so} \quad \sum_{r=1}^{k+1} r^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2 \\ &= (k+1) \left(\left(\frac{k}{6} \right) (2k+1) + (k+1) \right) \\ &= \frac{(k+1)}{6} (2k^2 + 7k + 6) \\ &= \frac{(k+1)}{6} ((k+1) + 1)(2(k+1) + 1) \end{aligned}$$

This matches the formula with n replaced by $k+1$.

Take out any common factors as soon as you can as you work – not a good idea to expand and simplify fully before looking for factors

If the statement is true for $n = k$ then it is also true for $n = k+1$.
As the statement is true for $n = 1$ it must be true for all $n \geq 1$

Multiples

Example Prove by induction that $9^n - 1$ is a multiple of 8

Step 1 : Check for $n=1$ $9^1 - 1 = 8$ which is a multiple of 8

Step 2 : Assume that the result holds for a value $n = k$

Then $9^k - 1 = 8m$ where m is an integer

Step 3 : Show that the results holds for $k + 1$

$$\begin{aligned} 9^{k+1} - 1 &= 9^k \times 9 - 1 \quad \text{but} \quad 9^k = 8m + 1 \\ \text{so } 9^{k+1} - 1 &= (8m + 1) \times 9 - 1 \\ &= 72m + 9 - 1 \\ &= 8(9m + 1) \quad \text{so } 9^{k+1} - 1 \text{ is also a multiple of 8} \end{aligned}$$

If the statement is true for $n = k$ then it is also true for $n = k+1$.
As the statement is true for $n = 1$ it must be true for all $n \geq 1$

2. Complex Numbers

$$z = a + ib$$

real imaginary

$i = \sqrt{-1} \qquad i^2 = -1$

- Addition and Subtraction**

$$(2 + 3i) + (5 - 2i) = 7 + i \quad (\text{add/subtract real part then imaginary part})$$

- Multiplication** - multiply out the same way you would $(x - 2)(x + 4)$

EXAMPLE $(2 - 3i)(6 + 2i) = 12 + 4i - 18i - 6i^2$

$$= 12 - 14i + 6$$

$$= 18 - 14i$$

- Complex Conjugate z^***

If $z = a + ib$ then its complex conjugate is $z^* = a - ib$

always collect the 'real' and 'imaginary' parts before looking for the conjugate

- Solving Equations** - if two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

EXAMPLE Find z when $5z - 2z^* = 3 - 14i$

Let $z = x + iy$ and so $z^* = x - iy$

$$5(x + iy) - 2(x - iy) = 3 - 14i$$

$$3x + 7iy = 3 - 14i$$

Equating real : $3x = 3$ so $x = 1$

Equating imaginary : $7y = -14$ so $y = -2$ $z = 1 - 2i$

- Quadratic equations – $ax^2 + bx + c = 0$** (where a, b and c are real)

If $b^2 - 4ac < 0$ then the equation has 2 complex roots which will be **complex conjugates**

EXAMPLE Solve $4x^2 + 2x + 1 = 0$ $a = 4$ $b = 2$ $c = 1$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 4 \times 1}}{2 \times 4}$$

$$x = \frac{-2 \pm \sqrt{-12}}{8} \qquad x = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$$

- Cubic equations – $ax^3 + bx^2 + cx + d = 0$**

3 real roots

1 real root and 1 pair of complex conjugate roots

EXAMPLE

The cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots $x = 2$ and $x = 1 - 2i$ and a third root. Find the value of the constants a, b, c and d

Third root is $1 + 2i$ (complex conjugate)

$$(x - 2)(x - (1 - 2i))(x - (1 + 2i)) = ax^3 + bx^2 + cx + d$$

$$(x - 2)(x^2 - 2x + 5)$$

$$= x^3 - 2x^2 + 5x - 2x^2 + 4x - 10$$

$$= x^3 - 4x^2 + 9x + 10$$

$$a = 1, b = -4, c = 9, d = 10$$

$$(x - 1 + 2i)(x - 1 - 2i)$$

$$= x^2 - x - 2xi - x + 1 + 2i + 2xi - 2i + 4$$

$$= x^2 - 2x + 5$$

- Quartic equations – $ax^4 + bx^3 + cx^2 + dx + e = 0$**

4 real roots

2 real roots and 1 pair of complex conjugate roots

2 pair of complex conjugate roots

- **ARGAND DIAGRAMS**

$$z = x + iy$$

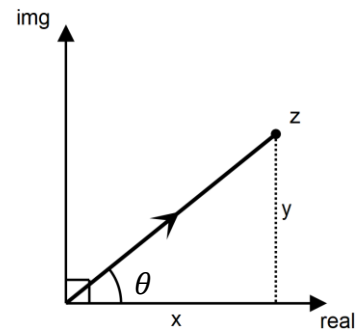
$$|z| = \sqrt{x^2 + y^2}$$

$$\arg(z) = \theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$\arg$ usually in radians where $-\pi < \theta < \pi$

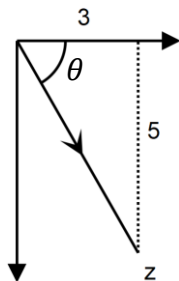
anti-clockwise from x axis gives a positive argument



- **Modulus-Argument form** $z = r(\cos \theta + i \sin \theta)$ where $r = |z|$ and $\theta = \arg z$

EXAMPLE

Write the complex number $z = 3 - 5i$ in modulus-argument form (radians to 2 d.p)



$$r = \sqrt{3^2 + 5^2}$$

$$= \sqrt{34}$$

$$\theta = \tan^{-1}\left(\frac{5}{3}\right)$$

$$\theta = -1.03$$

$$z = \sqrt{34}(\cos(-1.03) + i \sin(-1.03))$$

- **Multiplying and dividing complex numbers**

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

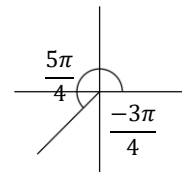
Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

You may need to adjust the arg to ensure that it lies within the range $-\pi < \theta < \pi$

$$\text{If } \theta_1 + \theta_2 = \frac{5\pi}{4}$$

$$= \frac{-3\pi}{4}$$



- **LOCI**

Circles

$|z| = a$ circle radius a centre at the origin

$|z - z_1| = a$ circle radius a centre at z_1

$$|z - 5 - 3i| = 4$$

$$|z - (5 + 3i)| = 4 \quad \text{Centre } 5 + 3i \quad \text{radius } 4$$

Half Lines - straight lines which start at a specific point and only have one direction.

$\arg z = \theta$ represents the **half** line starting at 0 inclined at an angle θ to the positive x direction

$\arg(z - z_1) = \theta$ represents the **half** line starting at z_1 inclined at an angle θ to the positive x direction

EXAMPLE

For $\arg(z - 3 - 2i) = \frac{3\pi}{4}$ construct the locus of z

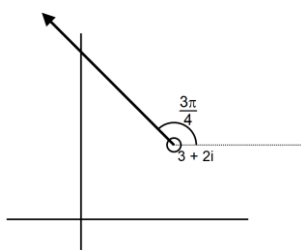
$$z = x + iy \quad \arg(z - (3 + 2i)) = \frac{3\pi}{4} \quad \arg(x + iy - (3 + 2i)) = \frac{3\pi}{4}$$

$$\tan^{-1}\left(\frac{y-2}{x-3}\right) = \frac{3\pi}{4}$$

$$\frac{y-2}{x-3} = -1$$

$$x + y = 5$$

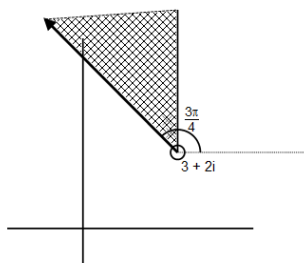
Restricted to $x < 3$ and $y > 2$



Regions

If the complex locus is an inequality then it will describe a region

$$\frac{\pi}{2} \leq \arg(z - 3 - 2i) \leq \frac{3\pi}{4}$$



3. Matrices

- **Order of a matrix** 'm x n' where m is the number of rows and n the number of columns
(If order is correct for a particular operation then matrices are described as **conformable**)

Zero Matrix – all elements are 0

Identity matrix $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

Transpose of a matrix $\mathbf{A}^T$ write the rows as columns and the columns as rows $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Special result $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

Determinant – every square matrix has a determinant

- if determinant is zero the matrix is **singular**

2 x 2 $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det \mathbf{M} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ $\det \mathbf{M}$ is the area scale factor

3 x 3 $\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\det \mathbf{M} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$\det \mathbf{M}$ is the volume scale factor

Adding a scalar multiple of a row/column to another row/column does not change the determinant of the matrix

Factorising the determinant

Works in same way as factor theorem if $(x - a)$ is a factor of $|M|$ then $|M| = 0$ when $x = a$

EXAMPLE : Factorise the determinant of $\begin{bmatrix} a & b & c \\ bc & ca & ab \\ b+c & c+a & a+b \end{bmatrix}$

$$C1 = C1 - C2 \quad \begin{vmatrix} a-b & b & c \\ bc-ca & ca & ab \\ b-a & c+a & a+b \end{vmatrix} = (b-a) \begin{vmatrix} -1 & b & c \\ c & ca & ab \\ 1 & c+a & a+b \end{vmatrix}$$

$$C2 = C2 - C3 \quad (b-a) \begin{vmatrix} -1 & b-c & c \\ c & a(c-b) & ab \\ 1 & c-b & a+b \end{vmatrix} = (b-a)(c-b) \begin{vmatrix} -1 & -1 & c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix}$$

$$R1 = R1 + R3 \quad (b-a)(c-b) \begin{vmatrix} 0 & 0 & a+b+c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix} = (b-a)(c-b)(c-a)(a+b+c)$$

Inverses $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$ (If a matrix is singular $\det \mathbf{A} = 0$ then it is not invertible (no inverse))

$$2 \times 2 \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{the inverse } \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$3 \times 3 \quad \mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{the inverse } \mathbf{M}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} A & -D & G \\ -B & E & -H \\ C & -F & I \end{bmatrix}$$

$$\text{where } A = \begin{vmatrix} e & f \\ g & h \end{vmatrix} \quad B = \begin{vmatrix} d & f \\ g & i \end{vmatrix} \quad C = \begin{vmatrix} d & e \\ g & h \end{vmatrix} \quad D = \begin{vmatrix} b & c \\ g & i \end{vmatrix} \quad E = \begin{vmatrix} a & c \\ g & i \end{vmatrix} \dots$$

4. Polynomials

Quadratics

For a quadratic equation $\mathbf{ax^2 + bx + c = 0}$ with roots α and β

$$\text{Sum of the roots } \alpha + \beta = -\frac{b}{a} \quad \text{Product of roots } \alpha\beta = \frac{c}{a}$$

Once the value $\alpha + \beta$ and $\alpha\beta$ have been found, new quadratic equations can be formed with roots :

Roots	Sum of roots	Product of roots
α^2 and β^2	$(\alpha + \beta)^2 - 2\alpha\beta$	$(\alpha\beta)^2$
α^3 and β^3	$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$	$(\alpha\beta)^3$
$\frac{1}{\alpha}$ and $\frac{1}{\beta}$	$\frac{\alpha + \beta}{\alpha\beta}$	$\frac{1}{\alpha\beta}$

The new equation becomes $\mathbf{x^2 - (sum of new roots)x + (product of new roots) = 0}$



EXAMPLE

The roots of the quadratic equation $3x^2 + 4x - 1 = 0$ are α and β

Determine a quadratic equation with integer coefficients which has roots $\alpha^3\beta$ and $\alpha\beta^3$

$$\text{Step 1 : } \alpha + \beta = -\frac{4}{3} \quad \text{and } \alpha\beta = -\frac{1}{3}$$

$$\alpha + \beta = -\frac{4}{3} \quad \alpha\beta = -\frac{1}{3}$$

$$\begin{aligned} \text{Step 2 : Sum of new roots } \alpha^3\beta + \alpha\beta^3 &= \alpha\beta(\alpha^2 + \beta^2) \\ &= -\frac{1}{3}((\alpha + \beta)^2 - 2\alpha\beta) \\ &= -\frac{1}{3}\left(\left(\frac{16}{9}\right) + \frac{2}{3}\right) \\ &= -\frac{22}{27} \end{aligned}$$

$$\begin{aligned} \text{Step 3: Product of new roots } \alpha^3\beta\alpha\beta^3 &= (\alpha\beta)^4 \\ &= \frac{1}{81} \end{aligned}$$

Don't forget the "= 0"

Step 4 : Form the new equation $x^2 + \frac{22}{27}x + \frac{1}{81} = 0$ $81x^2 + 66x + 1 = 0$

CUBIC EQUATIONS $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ

$$\alpha + \beta + \gamma = -\frac{b}{a} \quad \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} \quad \alpha\beta\gamma = -\frac{d}{a}$$

QUARTIC EQUATIONS (and beyond....) $ax^4 + bx^3 + cx^2 + dx + e = 0$

$$\sum \alpha = -\frac{b}{a} \quad \sum \alpha\beta = \frac{c}{a} \quad \sum \alpha\beta\gamma = -\frac{d}{a} \quad \sum \alpha\beta\gamma\delta = \frac{e}{a}$$

EXAMPLE

The quartic equation $x^4 + 2x^3 + 14x + 15 = 0$ has one root equal to $1 + 2i$. Find the other three roots.

As the coefficients are real, it follows that $1 - 2i$ is also a root (complex conjugate)

Hence $(x - (1 - 2i))(x - (1 + 2i))$ is a quadratic factor of the quartic

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + ax + b)$$

Equating coefficients: $2 = a - 2$

$$a = 4 \quad 15 = 5b$$

$$b = 3$$

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + 4x + 3)$$

the four roots are $1 + 2i$, $1 - 2i$, -3 , and -1 .

5. Series

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

Key Points

$$\sum_{r=1}^n 1 = n$$

$$\sum 5r^2 = 5 \sum r^2$$

Always look carefully at the limits

$$\sum_{r=n+1}^{2n} r^3 = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^n r^3$$

The method of differences

- Often seen as a 2 part question – i) show that ii) hence find
- ALWAYS show clearly the cancelling process

Example

Express $\frac{1}{r(r+2)}$ in partial fractions.

Hence show that $\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$

$$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$$

$$A(r+2) + Br = 1$$

$$A = \frac{1}{2} \quad B = -\frac{1}{2}$$

$$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

$$\begin{aligned}
\sum_{r=1}^n \frac{1}{r(r+2)} &= \frac{1}{2(1)} - \frac{1}{2(3)} + \frac{1}{2(2)} - \frac{1}{2(4)} + \frac{1}{2(3)} - \frac{1}{2(5)} + \frac{1}{2(4)} - \frac{1}{2(6)} + \dots + \frac{1}{2(n-1)} - \frac{1}{2(n+1)} + \frac{1}{2(n)} - \frac{1}{2(n+2)} \\
&= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \\
&= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)} \\
&= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)} \\
&= \frac{n(3n+5)}{4(n+1)(n+2)}
\end{aligned}$$

Maclaurin Series standard series given in the formula booklet

Example

Find the Maclaurin expansion of $f(x) = \ln(1 + 2x)$ up to the x^3 term.

State the range of values for which it is valid

$$\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} \\
\ln(1+2x) &= (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} \\
&= 2x - 2x^2 + \frac{8x^3}{3}
\end{aligned}$$

$$\ln(1+x) \text{ valid for } -1 < x < 1 \quad \ln(1+2x) \text{ valid for } -1 < 2x < 1 \quad \text{so } -\frac{1}{2} < x < \frac{1}{2}$$

Deriving a Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} \dots \dots \dots$$

Assumption

- $f(x)$ can be expressed as $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \dots \dots$
- The series can be differentiated term by term
- The function $f(x)$ and all of its derivatives exist at $x = 0$

If you are finding products of expansions check the range of values for which each is valid

6. Inequalities

Polynomials

Rearrange to the form $f(x) < > 0$

Identify the roots (critical values) $f(x) = 0$

Sketch the graph to identify the ranges of x for which the inequality is satisfied

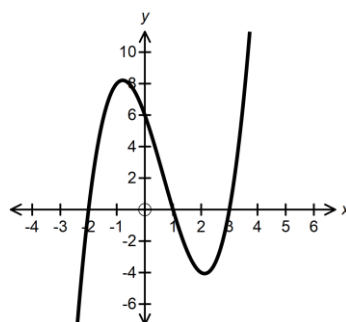
EXAMPLE

$$\text{Solve } x^3 - 2x^2 - 5x + 6 > 0$$

$$x^3 - 2x^2 - 5x + 6 = 0$$

$$(x-1)(x-3)(x+2) = 0$$

$$\{x \in \mathbb{R}: -2 < x < 1\} \cup \{x \in \mathbb{R}: x > 3\}$$



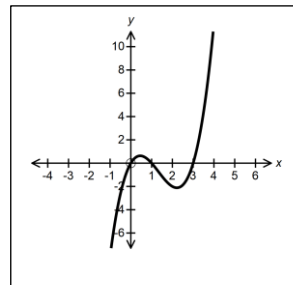
Algebraic Fractions

Multiply through by the square of each denominator and solve as above

EXAMPLE

$$\begin{aligned}\text{Solve } \frac{2x}{x-1} > x & \quad 2x(x-1) > x(x-1)^2 \\ 2x^2 - 2x > x^3 - 2x^2 + x & \\ x^3 - 4x^2 + 3x < 0 & \\ x(x-3)(x-1) < 0 &\end{aligned}$$

$$\{x \in \mathbb{R} : x < 0\} \cup \{x \in \mathbb{R} : 1 < x < 3\}$$



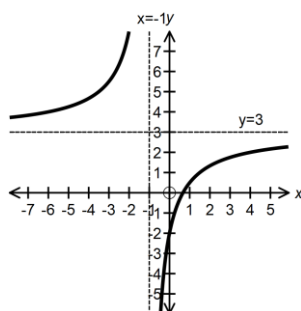
7. Graphs

Rational Functions

Linear denominator and numerator : $y = \frac{ax+b}{bx+c}$ 1 vertical asymptote and 1 horizontal asymptote

Example

Sketch the graph of $y = \frac{3x-2}{x+1}$



Vertical Asymptote : $x + 1 = 0 \quad x = -1$

Horizontal Asymptote :

divide all terms by the highest power of x

$$\frac{3 - \frac{2}{x}}{1 + \frac{1}{x}} \quad \text{as } x \rightarrow \infty \quad \frac{3 - \frac{2}{x}}{1 + \frac{1}{x}} \rightarrow 3 \quad y = 3$$

Check axis intercepts $x = 0 \quad y = -2 \quad y = 0 \quad x = \frac{2}{3}$

Quadratic denominator and/or numerator

Find the asymptotes as above but the graph may cross the horizontal asymptote

If the denominator has no real roots then there is no vertical asymptote

A graph may have an **oblique asymptote** instead of a horizontal asymptote (see below)

EXAMPLE

Sketch the graph of $y = \frac{x^2+4x+4}{x+1}$

$x = 0 \quad y = 4 \quad y = 0 \quad x = -2$

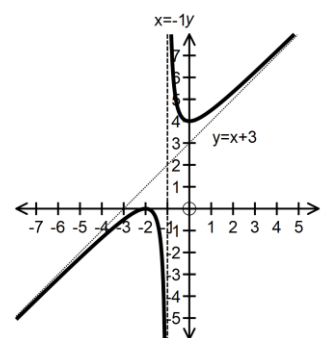
Vertical asymptote $x = -1$

Dividing through by x^2 results with an expression that is undefined as $x \rightarrow \infty$

Instead work out $(x^2 + 4x + 4) \div (x + 1)$

which gives $x + 3$ (and a remainder of 1)

The equation of the **oblique asymptote** is $y = x + 3$



Stationary points – using the discriminant

To find stationary points set $f(x) = k$ and rearrange to form an equation $ax^2 + bx + c = 0$

Calculate the discriminant

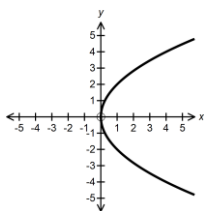
$b^2 - 4ac > 0$ the line $y = k$ intersects the graph twice

$b^2 - 4ac < 0$ the line $y = k$ does not intersect the graph

$b^2 - 4ac = 0$ the line touches the graph at a stationary point where $y = k$

Conic Sections

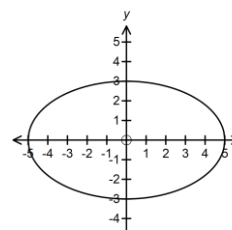
Parabolas $y^2 = 4ax$



Ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

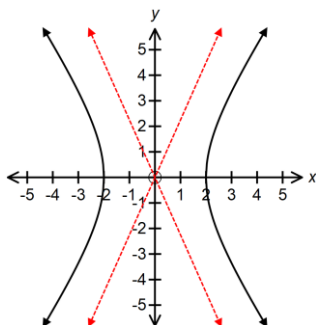
$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

$a > b$
major axis = $2a$
minor axis = $2b$



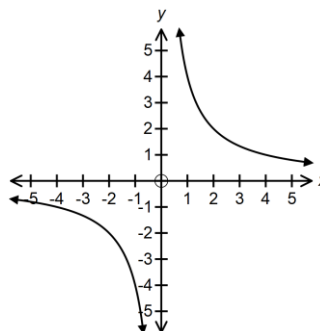
Hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{asymptotes } y = \pm \frac{b}{a}x$$



Rectangular Hyperbola (hyperbola where $a = b$)

$$xy = c^2$$



Transformations

Translation $\begin{bmatrix} a \\ b \end{bmatrix}$

Replace x with $(x - a)$

Replace y with $(y - b)$

Reflection in the line $y = x$

Replace x with y and vice versa

Stretch Parallel to the x -axis scale factor a Replace x with $\frac{x}{a}$

Stretch Parallel to the y -axis scale factor b Replace y with $\frac{y}{b}$

8. Integration

Volumes of revolution

Rotation about the x -axis $\int_{x=a}^{x=b} \pi y^2 dx$

Rotation about the y -axis $\int_{y=a}^{y=b} \pi x^2 dy$

Don't forget the π in your answer

Mean value of a Function

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$$

9. Vectors (3D)

$$xi + yj + zk \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Magnitude $|xi + yj + zk| = \sqrt{x^2 + y^2 + z^2}$

Straight lines **Vector Form** $r = a + \lambda b$ a : position vector b : direction vector (parallel to the line)

Cartesian form : $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ (x,y,z) is a general point on the line

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} \quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

EXAMPLE

Find the vector equation of the line with cartesian equation $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-5}{4}$

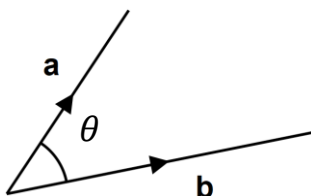
$$\mathbf{r} = (3\mathbf{i} - \mathbf{j} + 5\mathbf{k}) + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$$

Pairs of lines can - be parallel (look at direction vector)

- intersect - a solution exists when solving simultaneously for all 3 variables

- be skew (never meet) – no solution when solving simultaneously

Scalar product $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between vectors a and b when both directed away from the same point $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$



Perpendicular vectors have scalar product of zero

Finding the angle between 2 lines $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

EXAMPLE: Find the acute angle between the lines $\mathbf{r} = \begin{bmatrix} 0 \\ 7 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} -7 \\ 1 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$

Using the direction vectors

Let $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + 2\mathbf{k}$

$$\mathbf{a} \cdot \mathbf{b} = 4 \times -3 - 3 \times 0 + 1 \times 2 = -10 \quad |\mathbf{a}| = \sqrt{26} \quad |\mathbf{b}| = \sqrt{13}$$

$$\theta = \cos^{-1}\left(\frac{-10}{\sqrt{26}\sqrt{13}}\right) = 123^\circ \quad \text{Acute angle} = 57^\circ$$

Perpendicular distance from a point to a line - shortest path

EXAMPLE

Relative to a fixed origin O, the point A has position vector $(8\mathbf{i} + 13\mathbf{j} - 2\mathbf{k})$, the point B has position vector $(10\mathbf{i} + 14\mathbf{j} - 4\mathbf{k})$, and the point C has position vector $(9\mathbf{i} + 9\mathbf{j} + 6\mathbf{k})$.

Find the shortest distance from the point C to the line passing through A and B

$$\begin{aligned} \text{General point on line AB (point D)} &= (8\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \\ &= (8 + 2\lambda)\mathbf{i} + (13 + \lambda)\mathbf{j} + (-2 - 2\lambda)\mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{CD} &= (9 - (8 + 2\lambda))\mathbf{i} + (9 - (13 + \lambda))\mathbf{j} + (6 - (-2 - 2\lambda))\mathbf{k} \\ &= (1 - 2\lambda)\mathbf{i} + (-4 - \lambda)\mathbf{j} + (8 + 2\lambda)\mathbf{k} \end{aligned}$$

$$\text{Perpendicular if } (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (1 - 2\lambda)\mathbf{i} + (-4 - \lambda)\mathbf{j} + (8 + 2\lambda)\mathbf{k} = 0$$

$$2(1 - 2\lambda) + 1(-4 - \lambda) - 2(8 + 2\lambda) = 0$$

$$2 - 4\lambda - 4 - \lambda - 16 - 4\lambda = 0$$

$$-18 - 9\lambda = 0 \quad \lambda = -2$$

$$\text{When } \lambda = -2 \quad \mathbf{D} = 4\mathbf{i} + 11\mathbf{j} + 2\mathbf{k} \quad \mathbf{C} = 9\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}$$

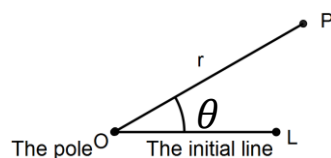
$$\begin{aligned} \text{Distance between C and D} &= \sqrt{(9 - 4)^2 + (9 - 11)^2 + (6 - 2)^2} \\ &= 3\sqrt{5} \end{aligned}$$

Perpendicular distance between parallel lines - pick a fixed point on one of the lines and complete as in the above example

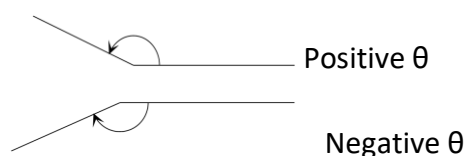
Shortest distance between 2 skew lines

$$\mathbf{r}_1 = \mathbf{a} + \lambda \mathbf{b} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{c} + \mu \mathbf{d} \quad \text{Shortest distance} = \left| \frac{(\mathbf{a}-\mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d})}{|\mathbf{b} \times \mathbf{d}|} \right|$$

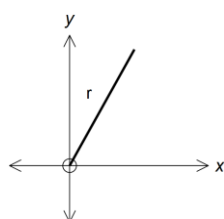
10. Polar Coordinates



P has polar coordinate (r, θ)



Polar – Cartesian



$$x = r \cos \theta \quad y = r \sin \theta$$

$$r^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}$$

Plotting Curves and Graphs

$r = a$ circle centre (0,0) radius a 	$\theta = \alpha$ Straight line (semi-infinite/half line) 	$r = 2a \cos \theta$ 	$r = 2a \sin \theta$
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Hints for sketching curves

- if r is expressed as a function of $\cos$ only - **symmetrical about the initial line**
- if r is expressed as a function of $\sin$ only - **symmetrical about the line $\theta = \frac{\pi}{2}$**
- if $r \rightarrow 0$ as $\theta \rightarrow \infty$ then the line $\theta = \alpha$ is a tangent to the curve at the pole
- **NEGATIVE VALUES** of r are not allowed – if $r < 0$ in the interval $\alpha < \theta < \beta$ then there is no curve in this interval

11. Hyperbolic Functions

You need to learn these as they are not given in the formula booklet

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

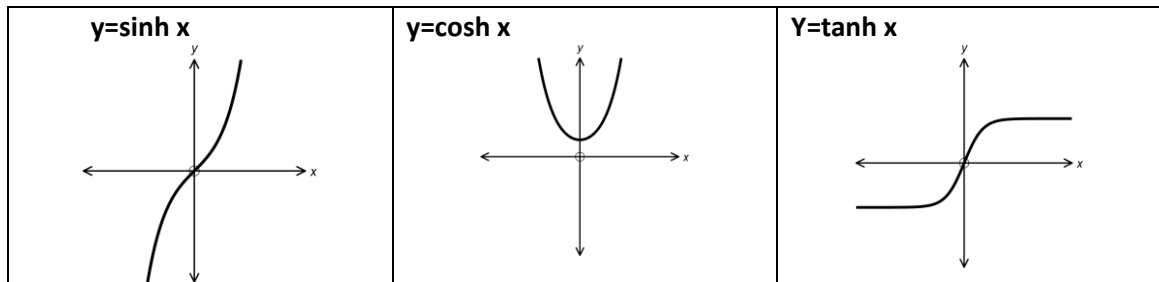
EXAMPLE Find the exact value of $\cosh(\ln 4)$

$$\cosh(\ln 4) = \frac{1}{2}(e^{\ln 4} + e^{-(\ln 4)})$$

$$= \frac{1}{2}\left(4 + \frac{1}{4}\right) = \frac{17}{8}$$

Identity : $\cosh^2 x - \sinh^2 x = 1$

Graphs of hyperbolic functions



Inverse hyperbolic functions – given in the formula booklet

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

$$x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

$$|x| < 1$$

EXAMPLE : Solve $\operatorname{artanh} x = \ln 4$

$$\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = \ln 4 \quad \ln \left(\frac{1+x}{1-x} \right) = \ln 16$$

$$\frac{1+x}{1-x} = 16$$

$$1 + x = 16 - 16x$$

$$x = \frac{15}{17}$$

Identities : $\coth^2 x - 1 = \operatorname{cosech}^2 x$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

EXAMPLE

Solve $2\coth^2 x - 5\operatorname{cosech} x = 5$
(answers in exact form)

Using $\coth^2 x = \operatorname{cosech} x + 1$

$$2(\operatorname{cosech} x + 1) - 5\operatorname{cosech} x = 5$$

$$2\operatorname{cosech} x + 2 - 5\operatorname{cosech} x - 5 = 0$$

$$2\operatorname{cosech} x - 5\operatorname{cosech} x - 3 = 0$$

$$(2\operatorname{cosech} x + 1)(\operatorname{cosech} x - 3) = 0$$

$$\operatorname{cosech} x = -\frac{1}{2} \quad \operatorname{cosech} x = 3$$

$$\sinh x = -2 \quad x = \sinh^{-1}(-2)$$

$$= \ln(-2 + \sqrt{(-2)^2 + 1})$$

$$= \ln(-2 + \sqrt{5})$$

$$\sinh x = \frac{1}{3} \quad x = \sinh^{-1}\left(\frac{1}{3}\right)$$

$$= \ln\left(\frac{1}{3} + \sqrt{\left(\frac{1}{3}\right)^2 + 1}\right)$$

$$= \ln\left(\frac{1}{3} + \sqrt{\frac{10}{9}}\right)$$