

1. Proof

Proof by Induction

- Step 1 : Show that the rule works for the first value e.g. $n = 1$
- Step 2 : Make a statement : *Assume that the result holds for $n = k$*
- Step 3 : Show that the results holds for $n = k + 1$

Summation of a series

Example Prove by induction that $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$

Step 1 : Check for $n=1$

$$\text{LHS : } 1^2 = 1 \quad \text{RHS : } \frac{1}{6} \times 2 \times 3 = 1$$

So the formula is true for $n=1$

Step 2 : Assume that the result holds for the k^{th} step, when $n=k$ then $\sum_{r=1}^k r^2 = \frac{k}{6}(k+1)(2k+1)$

Step 3 : Show that the results holds for $k + 1$

$$\begin{aligned} \sum_{r=1}^{k+1} r^2 &= \sum_{r=1}^k r^2 + (k+1)^2 \quad \text{so} \quad \sum_{r=1}^{k+1} r^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2 \\ &= (k+1) \left(\left(\frac{k}{6} \right) (2k+1) + (k+1) \right) \\ &= \frac{(k+1)}{6} (2k^2 + 7k + 6) \\ &= \frac{(k+1)}{6} ((k+1) + 1)(2(k+1) + 1) \end{aligned}$$

This matches the formula with n replaced by $k+1$.

Take out any common factors as soon as you can as you work – not a good idea to expand and simplify fully before looking for factors

If the statement is true for $n = k$ then it is also true for $n = k+1$.
As the statement is true for $n = 1$ it must be true for all $n \geq 1$

Multiples

Example Prove by induction that $9^n - 1$ is a multiple of 8

Step 1 : Check for $n=1$ $9^1 - 1 = 8$ which is a multiple of 8

Step 2 : Assume that the result holds for a value $n = k$

Then $9^k - 1 = 8m$ where m is an integer

Step 3 : Show that the results holds for $k + 1$

$$\begin{aligned} 9^{k+1} - 1 &= 9^k \times 9 - 1 \quad \text{but} \quad 9^k = 8m + 1 \\ \text{so } 9^{k+1} - 1 &= (8m + 1) \times 9 - 1 \\ &= 72m + 9 - 1 \\ &= 8(9m + 1) \quad \text{so } 9^{k+1} - 1 \text{ is also a multiple of 8} \end{aligned}$$

If the statement is true for $n = k$ then it is also true for $n = k+1$.
As the statement is true for $n = 1$ it must be true for all $n \geq 1$

2. Complex Numbers

$$z = a + ib$$

real imaginary

$i = \sqrt{-1} \qquad i^2 = -1$

- Addition and Subtraction**

$$(2 + 3i) + (5 - 2i) = 7 + i \qquad \text{(add/subtract real part then imaginary part)}$$

- Multiplication** - multiply out the same way you would $(x - 2)(x + 4)$

EXAMPLE $(2 - 3i)(6 + 2i) = 12 + 4i - 18i - 6i^2$

$$= 12 - 14i + 6$$

$$= 18 - 14i$$

- Complex Conjugate z^***

If $z = a + ib$ then its complex conjugate is $z^* = a - ib$

always collect the 'real' and 'imaginary' parts before looking for the conjugate

- Solving Equations** - if two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

EXAMPLE Find z when $5z - 2z^* = 3 - 14i$

Let $z = x + iy$ and so $z^* = x - iy$

$$5(x + iy) - 2(x - iy) = 3 - 14i$$

$$3x + 7iy = 3 - 14i$$

Equating real : $3x = 3$ so $x = 1$

Equating imaginary : $7y = -14$ so $y = -2$ $z = 1 - 2i$

- Quadratic equations – $ax^2 + bx + c = 0$** (where a, b and c are real)

If $b^2 - 4ac < 0$ then the equation has 2 complex roots which will be **complex conjugates**

EXAMPLE Solve $4x^2 + 2x + 1 = 0$ $a = 4$ $b = 2$ $c = 1$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 4 \times 1}}{2 \times 4}$$

$$x = \frac{-2 \pm \sqrt{-12}}{8} \qquad x = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$$

- Cubic equations – $ax^3 + bx^2 + cx + d = 0$**

3 real roots

1 real root and 1 pair of complex conjugate roots

EXAMPLE

The cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots $x = 2$ and $x = 1 - 2i$ and a third root. Find the value of the constants a, b, c and d

Third root is $1 + 2i$ (complex conjugate)

$$(x - 2)(x - (1 - 2i))(x - (1 + 2i)) = ax^3 + bx^2 + cx + d$$

$$(x - 2)(x^2 - 2x + 5)$$

$$= x^3 - 2x^2 + 5x - 2x^2 + 4x - 10$$

$$= x^3 - 4x^2 + 9x - 10$$

$$a = 1, b = -4, c = 9, d = -10$$

$$(x - 1 + 2i)(x - 1 - 2i)$$

$$= x^2 - x - 2xi - x + 1 + 2i + 2xi - 2i + 4$$

$$= x^2 - 2x + 5$$

- Quartic equations – $ax^4 + bx^3 + cx^2 + dx + e = 0$**

4 real roots

2 real roots and 1 pair of complex conjugate roots

2 pair of complex conjugate roots

- **ARGAND DIAGRAMS**

$$z = x + iy$$

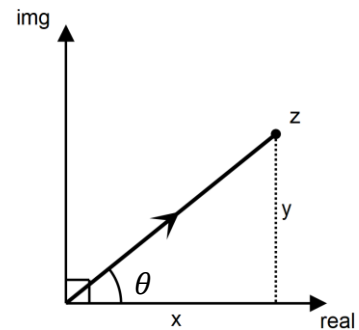
$$|z| = \sqrt{x^2 + y^2}$$

$$\arg(z) = \theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$\arg$ usually in radians where $-\pi < \theta < \pi$

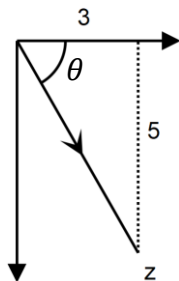
anti-clockwise from x axis gives a positive argument



- **Modulus-Argument form** $z = r(\cos \theta + i \sin \theta)$ where $r = |z|$ and $\theta = \arg z$

EXAMPLE

Write the complex number $z = 3 - 5i$ in modulus-argument form (radians to 2 d.p)



$$r = \sqrt{3^2 + 5^2}$$

$$= \sqrt{34}$$

$$\theta = \tan^{-1}\left(\frac{5}{3}\right)$$

$$\theta = -1.03$$

$$z = \sqrt{34}(\cos(-1.03) + i \sin(-1.03))$$

- **Multiplying and dividing complex numbers**

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

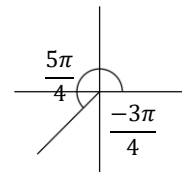
Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

You may need to adjust the arg to ensure that it lies within the range $-\pi < \theta < \pi$

$$\text{If } \theta_1 + \theta_2 = \frac{5\pi}{4}$$

$$= \frac{-3\pi}{4}$$



- **LOCI**

Circles

$|z| = a$ circle radius a centre at the origin

$|z - z_1| = a$ circle radius a centre at z_1

$$|z - 5 - 3i| = 4$$

$$|z - (5 + 3i)| = 4 \quad \text{Centre } 5 + 3i \quad \text{radius } 4$$

Half Lines - straight lines which start at a specific point and only have one direction.

$\arg z = \theta$ represents the **half** line starting at 0 inclined at an angle θ to the positive x direction

$\arg(z - z_1) = \theta$ represents the **half** line starting at z_1 inclined at an angle θ to the positive x direction

EXAMPLE

For $\arg(z - 3 - 2i) = \frac{3\pi}{4}$ construct the locus of z

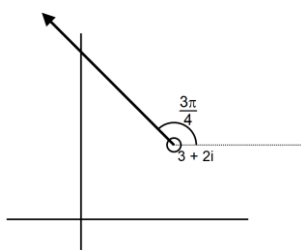
$$z = x + iy \quad \arg(z - (3 + 2i)) = \frac{3\pi}{4} \quad \arg(x + iy - (3 + 2i)) = \frac{3\pi}{4}$$

$$\tan^{-1}\left(\frac{y-2}{x-3}\right) = \frac{3\pi}{4}$$

$$\frac{y-2}{x-3} = -1$$

$$x + y = 5$$

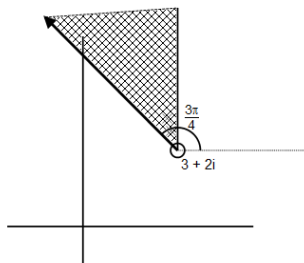
Restricted to $x < 3$ and $y > 2$



Regions

If the complex locus is an inequality then it will describe a region

$$\frac{\pi}{2} \leq \arg(z - 3 - 2i) \leq \frac{3\pi}{4}$$



3. Matrices

- **Order of a matrix** 'm x n' where m is the number of rows and n the number of columns
(If order is correct for a particular operation then matrices are described as **conformable**)

Zero Matrix – all elements are 0

Identity matrix $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

Transpose of a matrix $\mathbf{A}^T$ write the rows as columns and the columns as rows $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Special result $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

Determinant – every square matrix has a determinant
- if determinant is zero the matrix is **singular**

2 x 2 $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det \mathbf{M} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ $\det \mathbf{M}$ is the area scale factor

3 x 3 $\mathbf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\det \mathbf{M} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
 $\det \mathbf{M}$ is the volume scale factor

Adding a scalar multiple of a row/column to another row/column does not change the determinant of the matrix

Factorising the determinant

Works in same way as factor theorem if $(x - a)$ is a factor of $|M|$ then $|M| = 0$ when $x = a$

EXAMPLE : Factorise the determinant of $\begin{bmatrix} a & b & c \\ bc & ca & ab \\ b+c & c+a & a+b \end{bmatrix}$

$$C1 = C1 - C2 \quad \begin{vmatrix} a-b & b & c \\ bc-ca & ca & ab \\ b-a & c+a & a+b \end{vmatrix} = (b-a) \begin{vmatrix} -1 & b & c \\ c & ca & ab \\ 1 & c+a & a+b \end{vmatrix}$$

$$C2 = C2 - C3 \quad (b-a) \begin{vmatrix} -1 & b-c & c \\ c & a(c-b) & ab \\ 1 & c-b & a+b \end{vmatrix} = (b-a)(c-b) \begin{vmatrix} -1 & -1 & c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix}$$

$$R1 = R1 + R3 \quad (b-a)(c-b) \begin{vmatrix} 0 & 0 & a+b+c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix} = (b-a)(c-b)(c-a)(a+b+c)$$

Inverses $A \times A^{-1} = I$ (If a matrix is singular $\det A = 0$ then it is not invertible (no inverse))

$$2 \times 2 \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{the inverse } A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$3 \times 3 \quad M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{the inverse } M^{-1} = \frac{1}{\det A} \begin{bmatrix} A & -D & G \\ -B & E & -H \\ C & -F & I \end{bmatrix}$$

$$\text{where } A = \begin{vmatrix} e & f \\ g & h \end{vmatrix} \quad B = \begin{vmatrix} d & f \\ g & i \end{vmatrix} \quad C = \begin{vmatrix} d & e \\ g & h \end{vmatrix} \quad D = \begin{vmatrix} b & c \\ g & i \end{vmatrix} \quad E = \begin{vmatrix} a & c \\ g & i \end{vmatrix} \dots$$

Can be used to solve simultaneous equations

EXAMPLE Solve $x + 3y + 5z = -2$ $3x - 4y + 2z = 7$ $3x + 11y + 13z = 2$

$$AX = B \quad \begin{bmatrix} 1 & 3 & 5 \\ 3 & -4 & 2 \\ 3 & 11 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix} \quad A^{-1} = \frac{1}{52} \begin{bmatrix} -74 & 16 & 26 \\ -33 & -2 & 13 \\ 45 & -2 & -13 \end{bmatrix}$$

$$X = A^{-1}B \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{52} \begin{bmatrix} -74 & 16 & 26 \\ -33 & -2 & 13 \\ 45 & -2 & -13 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix} \quad x = 6 \quad y = 1.5 \quad z = -2.5 \quad \text{Unique solution}$$

Infinitely many solutions or no solution if $\det A = 0$

- Addition and Subtraction** – must have the same order

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 4 & 7 \end{bmatrix}$$

- Multiplication** – for AB number of columns in A must equal number of rows in B

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \times 2 + 2 \times 3 + 3 \times 4 & 1 \times -1 + 2 \times -2 + 3 \times 0 \\ 4 \times 2 + 5 \times 3 + 6 \times 4 & 4 \times -1 + 5 \times -2 + 6 \times 0 \end{bmatrix} \\ = \begin{bmatrix} 20 & -5 \\ 47 & -14 \end{bmatrix}$$

- Transformations– ORDER MATTERS**

Make sure you know the exact trig ratios

To calculate the coordinates of a point after a transformation

Multiply the Transformation Matrix by the coordinate

Find the position of point (2,1) after a stretch of Scale factor 5 parallel to the x-axis

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix} \quad (10,1)$$

To Identify a transformation from its matrix (consider the unit square)

Consider the points (1,0) and (0,1)

$$M = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{point (1,0)} \quad M \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \times 1 + 0 \times 0 \\ 0 \times 1 + 2 \times 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad (1,0) \rightarrow (4,0)$$

$$\text{point (0,1)} \quad M \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \times 0 + 0 \times 1 \\ 0 \times 0 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad (0,1) \rightarrow (0,2)$$

Stretch parallel to the x axis scale factor 4 and stretch parallel to the y axis scale factor 2

REFLECTIONS

In the y axis $M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

In the x-axis $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

In the line $y = x$ $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

In the line $y = (\tan\theta)x$ $M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

ENLARGEMENT

Scale factor k , centre $(0,0)$ $M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

STRETCH

Scale factor a parallel to the x-axis and Scale factor b parallel to the y-axis $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

ROTATION

Centre $(0,0)$ through θ **anti-clockwise** $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Centre $(0,0)$ through θ **clockwise** $M = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

ORDER MATTERS !!!! A figure is transformed by M_1 followed by M_2
Multiply $M_2 M_1$

WORKING IN 3D -same as in 2D – consider the points $(1,0,0)$ $(0,1,0)$ and $(0,0,1)$

Invariant Points - Points which are unaffected by a transformation and remain fixed

EXAMPLE Find the invariant points of the transformation with matrix $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad x + 2y = x \quad -y = y \quad (y = 0)$$

$y = 0$ the $x + 0 = x$ so all points $(x, 0)$ are invariant

Equation of the line of invariant points

The transformation matrix A is given by $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ find the equation of the line of invariant points passing through the

origin $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad x' = 2x + 3y \quad y' = x + 4y \quad y' = mx' + c$

$$x + 4y = m(2x + 3y) + c$$

$$x + 4(mx + c) = m(2x + 3(mx + c)) + c$$

$$x(3m^2 - 2m - 1) + c(3m - 3) = 0$$

$$x(m - 1)(3m + 1) + c(3m - 3) = 0$$

If $m = 1$ c can take any value

If $m = -\frac{1}{3}$ $c = 0$ (passing through the origin) $y = -\frac{1}{3}x$

4. Polynomials

Quadratics

For a quadratic equation $ax^2 + bx + c = 0$ with roots α and β

$$\text{Sum of the roots } \alpha + \beta = -\frac{b}{a} \quad \text{Product of roots } \alpha\beta = \frac{c}{a}$$

Once the value $\alpha + \beta$ and $\alpha\beta$ have been found, new quadratic equations can be formed with roots :

Roots	Sum of roots	Product of roots
α^2 and β^2	$(\alpha + \beta)^2 - 2\alpha\beta$	$(\alpha\beta)^2$
α^3 and β^3	$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$	$(\alpha\beta)^3$
$\frac{1}{\alpha}$ and $\frac{1}{\beta}$	$\frac{\alpha + \beta}{\alpha\beta}$	$\frac{1}{\alpha\beta}$

The new equation becomes $x^2 - (\text{sum of new roots})x + (\text{product of new roots}) = 0$



Don't forget the "= 0"

EXAMPLE

The roots of the quadratic equation $3x^2 + 4x - 1 = 0$ are α and β

Determine a quadratic equation with integer coefficients which has roots $\alpha^3\beta$ and $\alpha\beta^3$

$$\text{Step 1 : } \alpha + \beta = -\frac{4}{3} \quad \text{and} \quad \alpha\beta = -\frac{1}{3}$$

$$\alpha + \beta = -\frac{4}{3} \quad \alpha\beta = -\frac{1}{3}$$

$$\begin{aligned} \text{Step 2 : Sum of new roots } \alpha^3\beta + \alpha\beta^3 &= \alpha\beta(\alpha^2 + \beta^2) \\ &= -\frac{1}{3}((\alpha + \beta)^2 - 2\alpha\beta) \\ &= -\frac{1}{3}\left(\left(-\frac{4}{3}\right)^2 - 2\left(-\frac{1}{3}\right)\right) \\ &= -\frac{22}{27} \end{aligned}$$

$$\begin{aligned} \text{Step 3 : Product of new roots } \alpha^3\beta\alpha\beta^3 &= (\alpha\beta)^4 \\ &= \frac{1}{81} \end{aligned}$$

$$\text{Step 4 : Form the new equation } x^2 + \frac{22}{27}x + \frac{1}{81} = 0 \quad 81x^2 + 66x + 1 = 0$$

CUBIC EQUATIONS $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ

$$\alpha + \beta + \gamma = -\frac{b}{a} \quad \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} \quad \alpha\beta\gamma = -\frac{d}{a}$$

QUARTIC EQUATIONS (and beyond....) $ax^4 + bx^3 + cx^2 + dx + e = 0$

$$\sum \alpha = -\frac{b}{a} \quad \sum \alpha\beta = \frac{c}{a} \quad \sum \alpha\beta\gamma = -\frac{d}{a} \quad \sum \alpha\beta\gamma\delta = \frac{e}{a}$$

EXAMPLE

The quartic equation $x^4 + 2x^3 + 14x + 15 = 0$ has one root equal to $1 + 2i$. Find the other three roots.

As the coefficients are real, it follows that $1 - 2i$ is also a root (complex conjugate)

Hence $(x - (1 - 2i))(x - (1 + 2i))$ is a quadratic factor of the quartic

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + ax + b)$$

Equating coefficients: $2 = a - 2$

$$a = 4 \quad 15 = 5b$$

$$b = 3$$

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + 4x + 3)$$

the four roots are $1 + 2i$, $1 - 2i$, -3 , and -1 .

5. Series

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

Key Points

Always look carefully at the limits

$$\sum_{r=1}^n 1 = n$$

$$\sum_{r=n+1}^{2n} r^3 = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^n r^3$$

$$\sum 5r^2 = 5 \sum r^2$$

The method of differences

- Often seen as a 2 part question – i) show that ii) hence find
- ALWAYS show clearly the cancelling process

EXAMPLE

Express $\frac{1}{r(r+2)}$ in partial fractions.

Hence show that $\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$

$$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$$

$$A(r+2) + Br = 1$$

$$A = \frac{1}{2} \quad B = -\frac{1}{2}$$

$$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

$$\begin{aligned} \sum_{r=1}^n \frac{1}{r(r+2)} &= \frac{1}{2(1)} - \cancel{\frac{1}{2(3)}} + \frac{1}{2(2)} - \cancel{\frac{1}{2(4)}} + \cancel{\frac{1}{2(3)}} - \cancel{\frac{1}{2(5)}} + \cancel{\frac{1}{2(4)}} - \cancel{\frac{1}{2(7)}} \dots \dots \cancel{\frac{1}{2(n-1)}} - \frac{1}{2(n+1)} + \cancel{\frac{1}{2(n)}} - \frac{1}{2(n+2)} \\ &= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \\ &= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)} \\ &= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)} \\ &= \frac{n(3n+5)}{4(n+1)(n+2)} \end{aligned}$$

6. Calculus

Volumes of revolution

Rotation about the x – axis $\int_{x=a}^{x=b} \pi y^2 dx$

Rotation about the y – axis $\int_{y=a}^{y=b} \pi x^2 dy$

Parametric form

Rotation about the x – axis $\pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt} dt$

Rotation about the y – axis $\pi \int_{t_1}^{t_2} x^2 \frac{dy}{dt} dt$

Don't forget the π in your answer

EXAMPLE

Use the equation $x^2 + y^2 = r^2$ to show that the volume of sphere is $\frac{4\pi r^3}{3}$

$$y = \sqrt{r^2 - x^2} \quad \sqrt{r^2 - x^2} = 0 \quad x = \pm r$$

$$\begin{aligned} \text{Rotating about the x-axis} \quad & \int_{-r}^r \pi (\sqrt{r^2 - x^2})^2 dx \\ &= \pi \int_{-r}^r r^2 - x^2 dx \\ &= \pi \left[r^2 x - \frac{x^3}{3} \right] \\ &= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right] \\ &= \pi \left[r^3 - \frac{2r^3}{3} \right] \\ &= \frac{4}{3} \pi r^3 \end{aligned}$$

7. Vectors (3D) $xi + yj + zk \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Magnitude $|xi + yj + zk| = \sqrt{x^2 + y^2 + z^2}$

Straight lines **Vector Form** $r = a + \lambda b$ **a**: position vector **b**: direction vector (parallel to the line)

Cartesian form : $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ (x,y,z) is a general point on the line

$$\begin{aligned} r &= a + \lambda b & a &= a_1i + a_2j + a_3k \\ b &= b_1i + b_2j + b_3k \end{aligned}$$

EXAMPLE

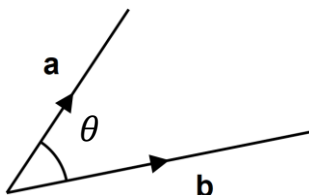
Find the vector equation of the line with cartesian equation $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-5}{4}$

$$r = (3i - j + 5k) + \lambda(2i + 3j + 4k)$$

Pairs of lines can - be parallel (look at direction vector)

- intersect - a solution exists when solving simultaneously for all 3 variables
- be skew (never meet) – no solution when solving simultaneously

Scalar product $a \cdot b = |a||b|\cos\theta$ where θ is the angle between vectors a and b when both directed away from the same point $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$



Perpendicular vectors have scalar product of zero

Finding the angle between 2 lines $\cos\theta = \frac{a \cdot b}{|a||b|}$

EXAMPLE: Find the acute angle between the lines $r = \begin{bmatrix} 0 \\ 7 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ and $r = \begin{bmatrix} -7 \\ 1 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$

Using the direction vectors

Let $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + 2\mathbf{k}$

$$\mathbf{a} \cdot \mathbf{b} = 4 \times -3 - 3 \times 0 + 1 \times 2 = -10 \quad |\mathbf{a}| = \sqrt{26} \quad |\mathbf{b}| = \sqrt{13}$$

$$\theta = \cos^{-1} \left(\frac{-10}{\sqrt{26}\sqrt{13}} \right) = 123^\circ \quad \text{Acute angle} = 57^\circ$$

Perpendicular distance from a point to a line - shortest path

EXAMPLE

Relative to a fixed origin O, the point A has position vector $(8\mathbf{i} + 13\mathbf{j} - 2\mathbf{k})$, the point B has position vector $(10\mathbf{i} + 14\mathbf{j} - 4\mathbf{k})$, and the point C has position vector $(9\mathbf{i} + 9\mathbf{j} + 6\mathbf{k})$.

Find the shortest distance from the point C to the line passing through A and B

$$\begin{aligned} \text{General point on line AB (point D)} &= (8\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \\ &= (8 + 2\lambda)\mathbf{i} + (13 + \lambda)\mathbf{j} + (-2 - 2\lambda)\mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{CD} &= (9 - (8 + 2\lambda))\mathbf{i} + (9 - (13 + \lambda))\mathbf{j} + (6 - (-2 - 2\lambda))\mathbf{k} \\ &= (1 - 2\lambda)\mathbf{i} + (-4 - \lambda)\mathbf{j} + (8 + 2\lambda)\mathbf{k} \end{aligned}$$

$$\text{Perpendicular if } (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (1 - 2\lambda)\mathbf{i} + (-4 - \lambda)\mathbf{j} + (8 + 2\lambda)\mathbf{k} = 0$$

$$\begin{aligned} 2(1 - 2\lambda) + 1(-4 - \lambda) - 2(8 + 2\lambda) &= 0 \\ 2 - 4\lambda - 4 - \lambda - 16 - 4\lambda &= 0 \\ -18 - 9\lambda &= 0 \quad \lambda = -2 \end{aligned}$$

$$\text{When } \lambda = -2 \quad \mathbf{D} = 4\mathbf{i} + 11\mathbf{j} + 2\mathbf{k} \quad \mathbf{C} = 9\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}$$

$$\begin{aligned} \text{Distance between C and D} &= \sqrt{(9 - 4)^2 + (9 - 11)^2 + (6 - 2)^2} \\ &= 3\sqrt{5} \end{aligned}$$

Perpendicular distance between parallel lines - pick a fixed point on one of the lines and complete as in the above example

Shortest distance between 2 skew lines

$$\mathbf{r}_1 = \mathbf{a} + \lambda \mathbf{b} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{c} + \mu \mathbf{d} \quad \text{Shortest distance} = \left| \frac{(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d})}{|\mathbf{b} \times \mathbf{d}|} \right|$$

Planes

Vector Form : $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$ a is a fixed position vector

b and c are non-parallel vectors in the plane λ and μ scalars

EXAMPLE

Find the vector equation of the plane that passes through points A(1,1,2) B(2, 0,3) and C(0,-2,1)

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$$

Scalar Form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = p$

r position vector of a general point on the plane

a position vector of a fixed point on the plane

n normal to the plane

p a scalar constant

Cartesian form

$$\mathbf{r} \cdot \mathbf{n} = p \quad ax + by + cz + d = 0$$

$$p = -d$$

EXAMPLE

Find the equation of the plane with normal vector $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ that contains point (3, 0, -2)

$$\mathbf{r} \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = 3 + 0 + 4 = 7 \quad x + 2y - 2z = 7$$

Angle between a line and a plane

Plane $\mathbf{r} \cdot \mathbf{n} = p$ and line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ $\sin \phi = \left| \frac{\mathbf{n} \cdot \mathbf{b}}{|\mathbf{n}| |\mathbf{b}|} \right|$

Angle between two planes

$\mathbf{n}_1$ is the normal to plane A $\mathbf{n}_2$ is the normal to plane B

The acute angle between the normals = the acute angle between the planes

Angle between the two planes is $180 - \theta$ where $\cos \theta = \left| \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right|$

EXAMPLE

Calculate the obtuse angle between the planes

$$\mathbf{r} \cdot (13\mathbf{i} - 9\mathbf{j} + 5\mathbf{k}) = 8$$

$$\mathbf{r} \cdot (2\mathbf{j} + 3\mathbf{k}) = 4$$

$$\mathbf{n}_1 = 13\mathbf{i} - 9\mathbf{j} + 5\mathbf{k} \quad \mathbf{n}_2 = 2\mathbf{j} + 3\mathbf{k}$$

$$\cos \theta = \left| \frac{(13\mathbf{i} - 9\mathbf{j} + 5\mathbf{k}) \cdot (2\mathbf{j} + 3\mathbf{k})}{\sqrt{13^2 + (-9)^2 + 5^2} \times \sqrt{2^2 + 3^2}} \right| = \left| \frac{-3}{\sqrt{275} \sqrt{13}} \right|$$

$$\theta = 87.1^\circ \text{ so the obtuse angle is } 92.9^\circ$$

Point of intersection of a line and a plane

EXAMPLE

Find the point of intersection of the line $\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$ and the plane $\mathbf{r} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 13$

$$\text{Line } \mathbf{r} = \begin{bmatrix} 1 + 4\lambda \\ 2 + \lambda \\ 1 - 3\lambda \end{bmatrix} \quad \begin{bmatrix} 1 + 4\lambda \\ 2 + \lambda \\ 1 - 3\lambda \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 13$$

$$3 + 12\lambda - 2 - \lambda + 2 - 6\lambda = 13$$

$$3 + 5\lambda = 13$$

$$\lambda = 2$$

$$\text{Point of intersection} = (9, 4, -5)$$

Intersecting Planes

Any two (non parallel planes) meet along a line – **line of intersection**

EXAMPLE

Find a cartesian equation of the line of intersection of the two planes $\mathbf{r} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 11$ and $\mathbf{r} \cdot \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = -7$

$$x - 2y + 3z = 11 \quad 2x + 2y - 2z = -7$$

$$(1) + (2) \quad 3x + z = 4 \quad z = 4 - 3x \quad x = \frac{4-z}{3}$$

$$\text{Subs into (2)} \quad 2x + 2y - 2(4 - 3x) = -7$$

$$8x + 2y = 1 \quad x = \frac{1-2y}{8}$$

$$x = \frac{1-2y}{8} = \frac{4-z}{3}$$

3 non-parallel planes can

- intersect at a single point
- all meet at the same line
- meet along different lines

Perpendicular distance between a point and a plane

Point $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and plane $\mathbf{r} \cdot \mathbf{n} = p$ Perpendicular distance = $\frac{|\mathbf{a} \cdot \mathbf{n} - p|}{|\mathbf{n}|}$

EXAMPLE

Find the shortest distance from point (1,4,1) to the plane $2x - y + z = 5$

Point = $\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ Normal to the plane = $2\mathbf{i} - \mathbf{j} + \mathbf{k}$

Shortest distance = perpendicular distance

$$\begin{aligned}\text{Distance} &= \frac{|2 \times 1 - 4 \times 1 + 1 \times 1 - 5|}{\sqrt{4 + 1 + 1}} \\ &= \frac{6}{\sqrt{6}} \\ &= \sqrt{6}\end{aligned}$$