

1. Proof

Proof by Induction

- Step 1 : Show that the rule works for the first value e.g. $n = 1$
- Step 2 : Make a statement : *Assume that the result holds for $n = k$*
- Step 3 : Show that the results holds for $n = k + 1$

Example Prove by induction that $9^n - 1$ is a multiple of 8

Step 1 : Check for $n=1$ $9^1 - 1 = 8$ which is a multiple of 8

Step 2 : *Assume that the result holds for a value $n = k$*

Then $9^k - 1 = 8m$ where m is an integer

Step 3 : Show that the results holds for $k + 1$

$$\begin{aligned} 9^{k+1} - 1 &= 9^k \times 9 - 1 & \text{but } 9^k &= 8m + 1 \\ \text{so } 9^{k+1} - 1 &= (8m + 1) \times 9 - 1 \\ &= 72m + 9 - 1 \\ &= 8(9m + 1) \end{aligned} \quad \text{so } 9^{k+1} - 1 \text{ is also a multiple of 8}$$

If the statement is true for $n = k$ then it is also true for $n = k+1$.

As the statement is true for $n = 1$ it must be true for all $n \geq 1$

2. Complex Numbers

$$z = \underset{\text{real}}{a} + \underset{\text{imaginary}}{ib}$$

$$i = \sqrt{-1} \quad i^2 = -1$$

Addition and Subtraction

$$(2 + 3i) + (5 - 2i) = 7 + i \quad (\text{add/subtract real part then imaginary part})$$

Multiplication - multiply out the same way you would $(x - 2)(x + 4)$

$$\begin{aligned} \text{EXAMPLE } (2 - 3i)(6 + 2i) &= 12 + 4i - 18i - 6i^2 \\ &= 12 - 14i + 6 \\ &= 18 - 14i \end{aligned}$$

Complex Conjugate z^*

If $z = a + ib$ then its complex conjugate is $z^* = a - ib$

always collect the 'real' and 'imaginary' parts before looking for the conjugate

Solving Equations - if two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

EXAMPLE Find z when $5z - 2z^* = 3 - 14i$

Let $z = x + iy$ and so $z^* = x - iy$

$$5(x + iy) - 2(x - iy) = 3 - 14i$$

$$3x + 7iy = 3 - 14i$$

$$\text{Equating real : } 3x = 3 \quad \text{so } x = 1$$

$$\text{Equating imaginary : } 7y = -14 \quad \text{so } y = -2 \quad z = 1 - 2i$$

Quadratic equations – $ax^2 + bx + c = 0$ (where a, b and c are real)

If $b^2 - 4ac < 0$ then the equation has 2 complex roots which will be **complex conjugates**

EXAMPLE Solve $4x^2 + 2x + 1 = 0$ $a = 4$ $b = 2$ $c = 1$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 4 \times 1}}{2 \times 4}$$

$$x = \frac{-2 \pm \sqrt{-12}}{8} \quad x = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$$

- Cubic equations – $ax^3 + bx^2 + cx + d = 0$**

3 real roots

1 real root and 1 pair of complex conjugate roots

EXAMPLE

The cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots $x = 2$ and $x = 1 - 2i$ and a third root. Find the value of the constants a, b, c and d

Third root is $1 + 2i$ (complex conjugate)

$$(x - 2)(x - (1 - 2i))(x - (1 + 2i)) = ax^3 + bx^2 + cx + d$$

$$\begin{aligned} (x - 2)(x^2 - 2x + 5) \\ = x^3 - 2x^2 + 5x - 2x^2 + 4x - 10 \\ = x^3 - 4x^2 + 9x + 10 \end{aligned}$$

$$a = 1, b = -4, c = 9, d = 10$$

$$\begin{aligned} (x - 1 + 2i)(x - 1 - 2i) \\ = x^2 - x - 2xi - x + 1 + 2i + 2xi - 2i + 4 \\ = x^2 - 2x + 5 \end{aligned}$$

- Quartic equations – $ax^4 + bx^3 + cx^2 + dx + e = 0$**

4 real roots

2 real roots and 1 pair of complex conjugate roots

2 pair of complex conjugate roots

- ARGAND DIAGRAMS**

$$z = x + iy$$

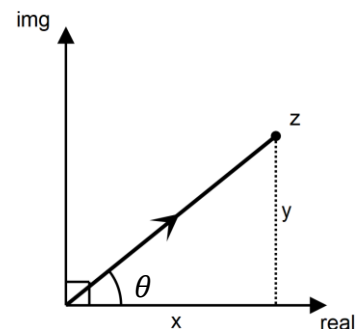
$$|z| = \sqrt{x^2 + y^2}$$

$$\arg(z) = \theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$\arg$ usually in radians where $-\pi < \theta < \pi$

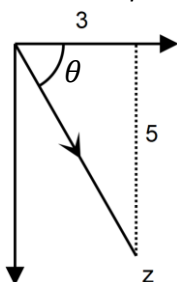
anti-clockwise from x axis gives a positive argument



- Modulus-Argument form** $z = r(\cos \theta + i \sin \theta)$ where $r = |z|$ and $\theta = \arg z$

EXAMPLE

Write the complex number $z = 3 - 5i$ in modulus-argument form (radians to 2 d.p)



$$\begin{aligned} r &= \sqrt{3^2 + 5^2} \\ &= \sqrt{34} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{5}{3}\right) \\ \theta &= -1.03 \end{aligned}$$

$$z = \sqrt{34}(\cos(-1.03) + i \sin(-1.03))$$

- **Multiplying and dividing complex numbers**

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

Multiplication

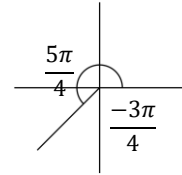
$$z_1 z_2 = r_1 r_2 (\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2))$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2))$$

You may need to adjust the arg to ensure that it lies within the range $-\pi < \theta < \pi$

$$\begin{aligned} \text{If } \theta_1 + \theta_2 &= \frac{5\pi}{4} \\ &= \frac{-3\pi}{4} \end{aligned}$$



- **LOC**

Circles

$|z| = a$ circle radius a centre at the origin

$|z - z_1| = a$ circle radius a centre at z_1

$$|z - 5 - 3i| = 4$$

$$|z - (5 + 3i)| = 4 \quad \text{Centre } 5 + 3i \quad \text{radius } 4$$

Half Lines - straight lines which start at a specific point and only have one direction.

$\arg z = \theta$ represents the **half** line starting at 0 inclined at an angle θ to the positive x direction

$\arg(z - z_1) = \theta$ represents the **half** line starting at z_1 inclined at an angle θ to the positive x direction

EXAMPLE

For $\arg(z - 3 - 2i) = \frac{3\pi}{4}$ construct the locus of z

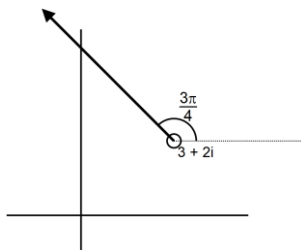
$$z = x + iy \quad \arg(z - (3 + 2i)) = \frac{3\pi}{4} \quad \arg(x + iy - (3 + 2i)) = \frac{3\pi}{4}$$

$$\tan^{-1} \left(\frac{y-2}{x-3} \right) = \frac{3\pi}{4}$$

$$\frac{y-2}{x-3} = -1$$

$$x + y = 5$$

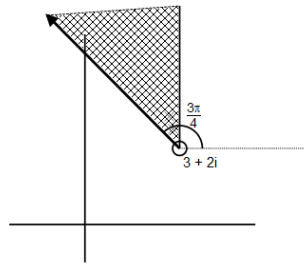
Restricted to $x < 3$ and $y > 2$



Regions

If the complex locus is an inequality then it will describe a region

$$\frac{\pi}{2} \leq \arg(z - 3 - 2i) \leq \frac{3\pi}{4}$$



3. Matrices

- **Order of a matrix** 'm x n' where m is the number of rows and n the number of columns
(If order is correct for a particular operation then matrices are described as **conformable**)

Zero Matrix – all elements are 0

Identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $AI = IA = I$

Transpose of a matrix A^T write the rows as columns and the columns as rows $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Special result $(AB)^T = B^T A^T$

Determinant – every square matrix has a determinant
- if determinant is zero the matrix is **singular**

2 x 2 $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det M = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ $\det M$ is the area scale factor

3 x 3 $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\det M = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
 $\det M$ is the volume scale factor

Adding a scalar multiple of a row/column to another row/column does not change the determinant of the matrix

Factorising the determinant

Works in same way as factor theorem if $(x - a)$ is a factor of $|M|$ then $|M| = 0$ when $x = a$

EXAMPLE : Factorise the determinant of $\begin{bmatrix} a & b & c \\ bc & ca & ab \\ b+c & c+a & a+b \end{bmatrix}$

$$C1 = C1 - C2 \quad \begin{vmatrix} a-b & b & c \\ bc-ca & ca & ab \\ b-a & c+a & a+b \end{vmatrix} = (b-a) \begin{vmatrix} -1 & b & c \\ c & ca & ab \\ 1 & c+a & a+b \end{vmatrix}$$

$$C2 = C2 - C3 \quad (b-a) \begin{vmatrix} -1 & b-c & c \\ c & a(c-b) & ab \\ 1 & c-b & a+b \end{vmatrix} = (b-a)(c-b) \begin{vmatrix} -1 & -1 & c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix}$$

$$R1 = R1 + R3 \quad (b-a)(c-b) \begin{vmatrix} 0 & 0 & a+b+c \\ c & a & ab \\ 1 & 1 & a+b \end{vmatrix} = (b-a)(c-b)(c-a)(a+b+c)$$

Inverses $A \times A^{-1} = I$ (If a matrix is singular $\det A = 0$ then it is not invertible (no inverse))

$$2 \times 2 \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{the inverse } A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$3 \times 3 \quad M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{the inverse } M^{-1} = \frac{1}{\det A} \begin{bmatrix} A & -D & G \\ -B & E & -H \\ C & -F & I \end{bmatrix}$$

$$\text{where } A = \begin{vmatrix} e & f \\ g & h \end{vmatrix} \quad B = \begin{vmatrix} d & f \\ g & i \end{vmatrix} \quad C = \begin{vmatrix} d & e \\ g & h \end{vmatrix} \quad D = \begin{vmatrix} b & c \\ g & i \end{vmatrix} \quad E = \begin{vmatrix} a & c \\ g & i \end{vmatrix} \dots$$

Can be used to solve simultaneous equations

EXAMPLE Solve $x + 3y + 5z = -2$ $3x - 4y + 2z = 7$ $3x + 11y + 13z = 2$

$$AX = B \quad \begin{bmatrix} 1 & 3 & 5 \\ 3 & -4 & 2 \\ 3 & 11 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix} \quad A^{-1} = \frac{1}{52} \begin{bmatrix} -74 & 16 & 26 \\ -33 & -2 & 13 \\ 45 & -2 & -13 \end{bmatrix}$$

$$X = A^{-1}B \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{52} \begin{bmatrix} -74 & 16 & 26 \\ -33 & -2 & 13 \\ 45 & -2 & -13 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix} \quad x = 6 \quad y = 1.5 \quad z = -2.5 \quad \text{Unique solution}$$

Infinitely many solutions or no solution if $\det A = 0$

- Addition and Subtraction** – must have the same order

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 4 & 7 \end{bmatrix}$$

- Multiplication** – for AB number of columns in A must equal number of rows in B

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \times 2 + 2 \times 3 + 3 \times 4 & 1 \times -1 + 2 \times -2 + 3 \times 0 \\ 4 \times 2 + 5 \times 3 + 3 \times 4 & 4 \times -1 + 5 \times -2 + 6 \times 0 \end{bmatrix} \\ = \begin{bmatrix} 20 & -5 \\ 47 & -14 \end{bmatrix}$$

- Transformations– ORDER MATTERS**

Make sure you know the exact trig ratios

To calculate the coordinates of a point after a transformation

Multiply the Transformation Matrix by the coordinate

Find the position of point (2,1) after a stretch of Scale factor 5 parallel to the x-axis

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix} \quad (10,1)$$

To Identify a transformation from its matrix (consider the unit square)

Consider the points (1,0) and (0,1)

$$M = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{point (1,0)} \quad M \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \times 1 + 0 \times 0 \\ 0 \times 1 + 2 \times 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad (1,0) \rightarrow (4,0)$$

point (0,1) $M \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \times 0 + 0 \times 1 \\ 0 \times 0 + 2 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad (0,1) \rightarrow (0,2)$

Stretch parallel to the x axis scale factor 4 and stretch parallel to the y axis scale factor 2
Standard Transformations

REFLECTIONS

In the y axis $M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

In the x-axis $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

In the line $y = x$ $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

In the line $y = (\tan \theta)x$ $M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

ENLARGEMENT

Scale factor k , centre (0,0) $M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

STRETCH

Scale factor a parallel to the x-axis and Scale factor b parallel to the y-axis $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

ROTATION

Centre (0,0) through θ **anti-clockwise** $M = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Centre (0,0) through θ **clockwise** $M = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

ORDER MATTERS !!!! A figure is transformed by M_1 followed by M_2
 Multiply $M_2 M_1$

WORKING IN 3D -same as in 2D – consider the points (1,0,0) (0,1,0) and (0,0,1)

Invariant Points - Points which are unaffected by a transformation and remain fixed

EXAMPLE Find the invariant points of the transformation with matrix $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad x + 2y = x \quad -y = y \quad (y = 0)$$

$y = 0$ the $x + 0 = x$ so all points $(x, 0)$ are invariant

Equation of the line of invariant points

The transformation matrix A is given by $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ find the equation of the line of invariant points passing through the

origin $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad x' = 2x + 3y \quad y' = x + 4y \quad y' = mx' + c$

$$x + 4y = m(2x + 3y) + c$$

$$x + 4(mx + c) = m(2x + 3(mx + c)) + c$$

$$x(3m^2 - 2m - 1) + c(3m - 3) = 0$$

$$x(m - 1)(3m + 1) + c(3m - 3) = 0$$

If $m = 1$ c can take any value If $m = -\frac{1}{3}$ $c = 0$ (passing through the origin) $y = -\frac{1}{3}x$

4. Vectors (2D & 3D) $xi + yj + zk$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Magnitude $|xi + yj + zk| = \sqrt{x^2 + y^2 + z^2}$

Straight lines Vector Form $r = a + \lambda b$ **a:** position vector **b:** direction vector (parallel to the line)

Cartesian form: $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ (x,y,z) is a general point on the line

$$r = a + \lambda b \quad a = a_1i + a_2j + a_3k$$

$$b = b_1i + b_2j + b_3k$$

EXAMPLE

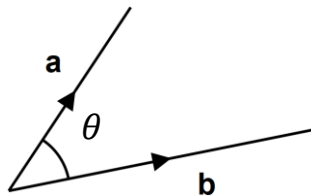
Find the vector equation of the line with cartesian equation $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-5}{4}$

$$r = (3i - j + 5k) + \lambda(2i + 3j + 4k)$$

Pairs of lines can

- be parallel (look at direction vector)
- intersect - a solution exists when solving simultaneously for all 3 variables
- be skew (never meet) - no solution when solving simultaneously

Scalar product $a \cdot b = |a||b|\cos\theta$ where θ is the angle between vectors a and b when both directed away from the same point $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$



Perpendicular vectors have scalar product of zero

Finding the angle between 2 lines $\cos\theta = \frac{a \cdot b}{|a||b|}$

EXAMPLE: Find the acute angle between the lines $r = \begin{bmatrix} 0 \\ 7 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ and $r = \begin{bmatrix} -7 \\ 1 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$

Using the direction vectors

Let $a = 4i - 3j + k$ and $b = -3i + 2k$

$$a \cdot b = 4 \times -3 - 3 \times 0 + 1 \times 2 = -10 \quad |a| = \sqrt{26} \quad |b| = \sqrt{13}$$

$$\theta = \cos^{-1}\left(\frac{-10}{\sqrt{26}\sqrt{13}}\right) = 123^\circ \quad \text{Acute angle} = 57^\circ$$

Perpendicular distance from a point to a line - shortest path

EXAMPLE

Relative to a fixed origin O, the point A has position vector $(8i + 13j - 2k)$, the point B has position vector $(10i + 14j - 4k)$, and the point C has position vector $(9i + 9j + 6k)$.

Find the shortest distance from the point C to the line passing through A and B

$$\begin{aligned} \text{General point on line AB (point D)} &= (8i + 13j - 2k) + \lambda(2i + j - 2k) \\ &= (8 + 2\lambda)i + (13 + \lambda)j + (-2 - 2\lambda)k \end{aligned}$$

$$CD = (9 - (8 + 2\lambda))i + (9 - (13 + \lambda))j + (6 - (-2 - 2\lambda))k$$

$$= (1 - 2\lambda)i + (-4 - \lambda)j + (8 + 2\lambda)k$$

Perpendicular if $(2i + j - 2k) \cdot (1 - 2\lambda)i + (-4 - \lambda)j + (8 + 2\lambda)k = 0$

$$\begin{aligned} 2(1 - 2\lambda) + 1(-4 - \lambda) - 2(8 + 2\lambda) &= 0 \\ 2 - 4\lambda - 4 - \lambda - 16 - 4\lambda &= 0 \\ -18 - 9\lambda &= 0 \quad \lambda = -2 \end{aligned}$$

When $\lambda = -2$ $D = 4i + 11j + 2k$ $C = 9i + 9j + 6k$

$$\begin{aligned} \text{Distance between C and D} &= \sqrt{(9 - 4)^2 + (9 - 11)^2 + (6 - 2)^2} \\ &= 3\sqrt{5} \end{aligned}$$

Perpendicular distance between parallel lines - pick a fixed point on one of the lines and complete as in the above example

Shortest distance between 2 skew lines

$$\mathbf{r}_1 = \mathbf{a} + \lambda \mathbf{b} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{c} + \mu \mathbf{d} \quad \text{Shortest distance} = \left| \frac{(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d})}{|\mathbf{b} \times \mathbf{d}|} \right|$$

$$x = \frac{1-2y}{8} = \frac{4-z}{3}$$

Vector Product

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta \hat{\mathbf{n}}$ θ is the angle between the 2 vectors when they are directed away from the same point

$\hat{\mathbf{n}}$ is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ (in a right-handed sense)

If 2 vectors are parallel then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

The vector product is the determinant of a 3 x 3 matrix

$$\mathbf{a} = a_1i + a_2j + a_3k \quad \mathbf{b} = a_1i + a_2j + a_3k$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

5. Polynomials

Quadratics

For a quadratic equation $\mathbf{ax^2 + bx + c = 0}$ with roots α and β

$$\text{Sum of the roots } \alpha + \beta = -\frac{b}{a} \quad \text{Product of roots } \alpha\beta = \frac{c}{a}$$

Once the value $\alpha + \beta$ and $\alpha\beta$ have been found, new quadratic equations can be formed with roots :

Roots	Sum of roots	Product of roots
α^2 and β^2	$(\alpha + \beta)^2 - 2\alpha\beta$	$(\alpha\beta)^2$
α^3 and β^3	$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$	$(\alpha\beta)^3$
$\frac{1}{\alpha}$ and $\frac{1}{\beta}$	$\frac{\alpha + \beta}{\alpha\beta}$	$\frac{1}{\alpha\beta}$

The new equation becomes $x^2 - (\text{sum of new roots})x + (\text{product of new roots}) = 0$

EXAMPLE

The roots of the quadratic equation $3x^2 + 4x - 1 = 0$ are α and β

Determine a quadratic equation with integer coefficients which has roots $\alpha^3\beta$ and $\alpha\beta^3$

Step 1 : $\alpha + \beta = -\frac{4}{3}$ and $\alpha\beta = -\frac{1}{3}$

$$\alpha + \beta = -\frac{4}{3} \quad \alpha\beta = -\frac{1}{3}$$

$$\begin{aligned} \text{Step 2 : Sum of new roots } \alpha^3\beta + \alpha\beta^3 &= \alpha\beta(\alpha^2 + \beta^2) \\ &= -\frac{1}{3}((\alpha + \beta)^2 - 2\alpha\beta) \\ &= -\frac{1}{3}\left(\left(-\frac{4}{3}\right)^2 - 2\left(-\frac{1}{3}\right)\right) \\ &= -\frac{1}{3}\left(\frac{16}{9} + \frac{2}{3}\right) \\ &= -\frac{22}{27} \end{aligned}$$

$$\begin{aligned} \text{Step 3 : Product of new roots } \alpha^3\beta\alpha\beta^3 &= (\alpha\beta)^4 \\ &= \frac{1}{81} \end{aligned}$$

$$\text{Step 4 : Form the new equation } x^2 + \frac{22}{27}x + \frac{1}{81} = 0 \quad 81x^2 + 66x + 1 = 0$$

CUBIC EQUATIONS $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ

$$\alpha + \beta + \gamma = -\frac{b}{a} \quad \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} \quad \alpha\beta\gamma = -\frac{d}{a}$$

QUARTIC EQUATIONS (and beyond....) $ax^4 + bx^3 + cx^2 + dx + e = 0$

$$\sum \alpha = -\frac{b}{a} \quad \sum \alpha\beta = \frac{c}{a} \quad \sum \alpha\beta\gamma = -\frac{d}{a} \quad \sum \alpha\beta\gamma\delta = \frac{e}{a}$$

EXAMPLE

The quartic equation $x^4 + 2x^3 + 14x + 15 = 0$ has one root equal to $1 + 2i$. Find the other three roots.

As the coefficients are real, it follows that $1 - 2i$ is also a root (complex conjugate)

Hence $(x - (1 - 2i))(x - (1 + 2i))$ is a quadratic factor of the quartic

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + ax + b)$$

Equating coefficients: $2 = a - 2$

$$a = 4 \quad 15 = 5b$$

$$b = 3$$

$$x^4 + 2x^3 + 14x + 15 \equiv (x^2 - 2x + 5)(x^2 + 4x + 3)$$

the four roots are $1 + 2i$, $1 - 2i$, -3 , and -1 .